# A First Decision Procedure for Almost-Sure Termination of Probabilistic Term Rewriting

**Jan-Christoph Kassing**[*1], Arion Scheid[*2], Henri Nagel[*1], and Jürgen Giesl[*1]
[*1] RWTH Aachen University, [*2] TU Vienna
VeriProb Workshop 2026

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}(\,[\,]\,) \to \mathsf{0} \qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}(\,[\,]\,) \to \mathsf{0} \qquad\qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$\mathsf{len}(\mathsf{a} : (\mathsf{b} : [\,]))$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}([\,]) \to \mathsf{0} \qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : [\,]))$$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}(\texttt{[]}) \to \mathsf{0} \qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : \texttt{[]})) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : \texttt{[]})) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{len}(\texttt{[]})))$$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}(\texttt{[]}) \to \mathsf{0} \qquad\qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : \texttt{[]})) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : \texttt{[]})) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{len}(\texttt{[]}))) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{0}))$$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}([\,]) \to \mathsf{0} \qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{len}([\,]))) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{0}))$$

## Termination

A TRS $\mathcal{R}$ is *terminating* if there is *no* infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}(\texttt{[]}) \to \mathsf{0} \qquad\qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : \texttt{[]})) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : \texttt{[]})) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{len}(\texttt{[]}))) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{0})) \quad \Longrightarrow \text{terminating}$$

## Termination

A TRS $\mathcal{R}$ is *terminating* if there is *no* infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}([\,]) \to \mathsf{0} \qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{len}([\,]))) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{0})) \implies \text{terminating}$$

## Termination

A TRS $\mathcal{R}$ is *terminating* if there is *no* infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$

$\mathcal{R}_{\mathsf{g}}$

$$\mathsf{g} \to \mathsf{c}(\mathsf{g}, \mathsf{g})$$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}([\,]) \to \mathsf{0} \qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{len}([\,]))) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{0})) \implies \text{terminating}$$

## Termination

A TRS $\mathcal{R}$ is *terminating* if there is *no* infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$

$\mathcal{R}_{\mathsf{g}}$

$$\mathsf{g} \to \mathsf{c}(\mathsf{g}, \mathsf{g})$$

$\mathsf{g}$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}([\,]) \to \mathsf{0} \qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{len}([\,]))) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{0})) \implies \text{terminating}$$

## Termination

A TRS $\mathcal{R}$ is *terminating* if there is *no* infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$

$\mathcal{R}_{\mathsf{g}}$

$$\mathsf{g} \to \mathsf{c}(\mathsf{g}, \mathsf{g})$$

$$\mathsf{g} \to_{\mathcal{R}_{\mathsf{g}}} \mathsf{c}(\mathsf{g}, \mathsf{g})$$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}([\,]) \to \mathsf{0} \qquad\qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{len}([\,]))) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{0})) \quad \Longrightarrow \text{terminating}$$

## Termination

A TRS $\mathcal{R}$ is *terminating* if there is *no* infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$

$\mathcal{R}_{\mathsf{g}}$

$$\mathsf{g} \to \mathsf{c}(\mathsf{g}, \mathsf{g})$$

$$\mathsf{g} \to_{\mathcal{R}_{\mathsf{g}}} \mathsf{c}(\mathsf{g}, \mathsf{g}) \to_{\mathcal{R}_{\mathsf{g}}} \mathsf{c}(\mathsf{c}(\mathsf{g}, \mathsf{g}), \mathsf{g})$$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}([\,]) \to \mathsf{0} \qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{len}([\,]))) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{0})) \quad \Longrightarrow \text{terminating}$$

## Termination

A TRS $\mathcal{R}$ is *terminating* if there is *no* infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$

$\mathcal{R}_{\mathsf{g}}$

$$\mathsf{g} \to \mathsf{c}(\mathsf{g}, \mathsf{g})$$

$$\mathsf{g} \to_{\mathcal{R}_{\mathsf{g}}} \mathsf{c}(\mathsf{g}, \mathsf{g}) \to_{\mathcal{R}_{\mathsf{g}}} \mathsf{c}(\mathsf{c}(\mathsf{g}, \mathsf{g}), \mathsf{g}) \to_{\mathcal{R}_{\mathsf{g}}} \ldots$$

# Term Rewrite Systems (TRS)

$\mathcal{R}_{\mathsf{len}}$

$$\mathsf{len}([\,]) \to \mathsf{0} \qquad \mathsf{len}(x : xs) \to \mathsf{s}(\mathsf{len}(xs))$$

(Apply a rule $\ell \to r$ at a matched redex)

$$\mathsf{len}(\mathsf{a} : (\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{len}(\mathsf{b} : [\,])) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{len}([\,]))) \to_{\mathcal{R}_{\mathsf{len}}} \mathsf{s}(\mathsf{s}(\mathsf{0})) \quad \Longrightarrow \text{terminating}$$

## Termination

A TRS $\mathcal{R}$ is *terminating* if there is *no* infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$

$\mathcal{R}_{\mathsf{g}}$

$$\mathsf{g} \to \mathsf{c}(\mathsf{g}, \mathsf{g})$$

$$\mathsf{g} \to_{\mathcal{R}_{\mathsf{g}}} \mathsf{c}(\mathsf{g}, \mathsf{g}) \to_{\mathcal{R}_{\mathsf{g}}} \mathsf{c}(\mathsf{c}(\mathsf{g}, \mathsf{g}), \mathsf{g}) \to_{\mathcal{R}_{\mathsf{g}}} \ldots \quad \Longrightarrow \text{not terminating}$$

# Probabilistic Term Rewrite Systems (PTRS)

$\mathcal{P}_{\mathsf{rw}}$ $$\mathrm{g} \to \{1/2 : \mathrm{c}(\mathrm{g}, \mathrm{g}),\ 1/2 : 0\}$$

# Probabilistic Term Rewrite Systems (PTRS)

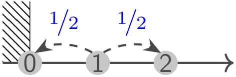
$\mathcal{P}_{\mathsf{rw}}$ $\qquad \mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g}, \mathsf{g}),\ 1/2 : \mathsf{0}\}$

Right-hand side is a multi-distribution $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$.

# Probabilistic Term Rewrite Systems (PTRS)

$\mathcal{P}_{\mathsf{rw}}$ $\qquad \mathsf{g} \to \{{}^1\!/\!{}_2 : \mathsf{c}(\mathsf{g},\mathsf{g}),\ {}^1\!/\!{}_2 : 0\}$

Right-hand side is a multi-distribution $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$.



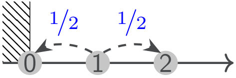
*symmetric random walk*

$$\{1 : \mathsf{g}\}$$

# Probabilistic Term Rewrite Systems (PTRS)

$\mathcal{P}_{\mathsf{rw}}$ $\quad \mathsf{g} \to \{\nicefrac{1}{2} : \mathsf{c}(\mathsf{g},\mathsf{g}),\ \nicefrac{1}{2} : \mathsf{0}\}$

Right-hand side is a multi-distribution $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$.



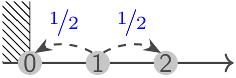
*symmetric random walk*

$$\begin{aligned} & \{1 : \mathsf{g}\} \\ \rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} \quad & \{\nicefrac{1}{2} : \mathsf{c}(\mathsf{g},\mathsf{g}), \nicefrac{1}{2} : \mathsf{0}, \} \end{aligned}$$

# Probabilistic Term Rewrite Systems (PTRS)

$\mathcal{P}_{\mathsf{rw}}$ $\qquad \mathsf{g} \to \{\nicefrac{1}{2} : \mathsf{c}(\mathsf{g}, \mathsf{g}),\ \nicefrac{1}{2} : \mathsf{0}\}$

Right-hand side is a multi-distribution $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$.



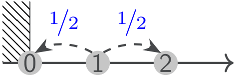

*symmetric random walk*

$$
\begin{aligned}
& \{1 : \mathsf{g}\} \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} \; & \{\nicefrac{1}{2} : \mathsf{c}(\mathsf{g}, \mathsf{g}), \nicefrac{1}{2} : \mathsf{0}, \} \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} \; & \{\nicefrac{1}{4} : \mathsf{c}(\mathsf{c}(\mathsf{g}, \mathsf{g}), \mathsf{g}), \nicefrac{1}{4} : \mathsf{c}(\mathsf{0}, \mathsf{g}), \nicefrac{1}{2} : \mathsf{0}\}
\end{aligned}
$$

# Probabilistic Term Rewrite Systems (PTRS)

$\mathcal{P}_{\mathsf{rw}}$ $$\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g}, \mathsf{g}),\ 1/2 : \mathsf{0}\}$$

Right-hand side is a multi-distribution $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$.



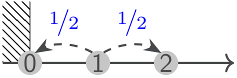
*symmetric random walk*

$$\begin{aligned} & \{1 : \mathsf{g}\} \\ \rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \{1/2 : \mathsf{c}(\mathsf{g}, \mathsf{g}), 1/2 : \mathsf{0}, \} \\ \rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \{1/4 : \mathsf{c}(\mathsf{c}(\mathsf{g}, \mathsf{g}), \mathsf{g}), 1/4 : \mathsf{c}(\mathsf{0}, \mathsf{g}), 1/2 : \mathsf{0}\} \\ \rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \quad \ldots \end{aligned}$$

# Probabilistic Term Rewrite Systems (PTRS)

$\mathcal{P}_{\mathsf{rw}}$ $\qquad \mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g},\mathsf{g}),\ 1/2 : \mathsf{0}\}$

Right-hand side is a multi-distribution $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$.



*symmetric random walk*

$$
\begin{aligned}
& \{1 : \mathsf{g}\} \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} \; & \{1/2 : \mathsf{c}(\mathsf{g},\mathsf{g}), 1/2 : \mathsf{0}, \} \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} \; & \{1/4 : \mathsf{c}(\mathsf{c}(\mathsf{g},\mathsf{g}),\mathsf{g}), 1/4 : \mathsf{c}(\mathsf{0},\mathsf{g}), 1/2 : \mathsf{0}\} \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} \; & \ldots
\end{aligned}
$$

## Almost-Sure Termination (AST)

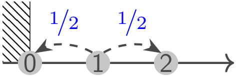
$|\mu_n|$ = probability mass on normal forms in $\mu_n$.

$\mathcal{P}$ is AST iff $\lim\limits_{n\to\infty} |\mu_n| = 1$ for every sequence $\mu_0 \rightrightarrows_{\mathcal{P}} \mu_1 \rightrightarrows_{\mathcal{P}} \cdots$

# Probabilistic Term Rewrite Systems (PTRS)

$\mathcal{P}_{\mathsf{rw}}$ $\quad\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g}, \mathsf{g}),\ 1/2 : \mathsf{0}\}$

Right-hand side is a multi-distribution $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$.



*symmetric random walk*

$$
\begin{array}{lll}
 & \{1 : \mathsf{g}\} & |\mu_0| = 0 \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \{1/2 : \mathsf{c}(\mathsf{g}, \mathsf{g}), 1/2 : \mathsf{0}, \} & \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \{1/4 : \mathsf{c}(\mathsf{c}(\mathsf{g}, \mathsf{g}), \mathsf{g}), 1/4 : \mathsf{c}(\mathsf{0}, \mathsf{g}), 1/2 : \mathsf{0}\} & \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \ldots &
\end{array}
$$

## Almost-Sure Termination (AST)

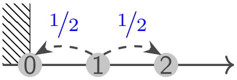
$|\mu_n|$ = probability mass on normal forms in $\mu_n$.

$\mathcal{P}$ is AST iff $\lim\limits_{n \to \infty} |\mu_n| = 1$ for every sequence $\mu_0 \rightrightarrows_{\mathcal{P}} \mu_1 \rightrightarrows_{\mathcal{P}} \cdots$

# Probabilistic Term Rewrite Systems (PTRS)

$\mathcal{P}_{\mathsf{rw}}$ $\qquad \mathsf{g} \to \{{}^{1}/_{2} : \mathsf{c}(\mathsf{g}, \mathsf{g}),\ {}^{1}/_{2} : \mathsf{0}\}$

Right-hand side is a multi-distribution $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$.



*symmetric random walk*

$$
\begin{array}{lll}
 & \{1 : \mathsf{g}\} & |\mu_0| = 0 \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \{{}^{1}/_{2} : \mathsf{c}(\mathsf{g}, \mathsf{g}), {}^{1}/_{2} : \mathsf{0}, \} & |\mu_1| = {}^{1}/_{2} \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \{{}^{1}/_{4} : \mathsf{c}(\mathsf{c}(\mathsf{g}, \mathsf{g}), \mathsf{g}), {}^{1}/_{4} : \mathsf{c}(\mathsf{0}, \mathsf{g}), {}^{1}/_{2} : \mathsf{0}\} & \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \ldots &
\end{array}
$$

## Almost-Sure Termination (AST)

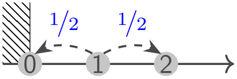
$|\mu_n|$ = probability mass on normal forms in $\mu_n$.

$\mathcal{P}$ is AST iff $\lim\limits_{n \to \infty} |\mu_n| = 1$ for every sequence $\mu_0 \rightrightarrows_{\mathcal{P}} \mu_1 \rightrightarrows_{\mathcal{P}} \cdots$

# Probabilistic Term Rewrite Systems (PTRS)

$\mathcal{P}_{\mathsf{rw}}$ $$\mathsf{g} \to \{{}^1\!/_2 : \mathsf{c}(\mathsf{g},\mathsf{g}),\ {}^1\!/_2 : \mathsf{0}\}$$

Right-hand side is a multi-distribution $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$.



*symmetric random walk*

$$\begin{array}{llr} & \{1 : \mathsf{g}\} & |\mu_0| = 0 \\ \rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \{{}^1\!/_2 : \mathsf{c}(\mathsf{g},\mathsf{g}), {}^1\!/_2 : \mathsf{0}, \} & |\mu_1| = {}^1\!/_2 \\ \rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \{{}^1\!/_4 : \mathsf{c}(\mathsf{c}(\mathsf{g},\mathsf{g}),\mathsf{g}), {}^1\!/_4 : \mathsf{c}(\mathsf{0},\mathsf{g}), {}^1\!/_2 : \mathsf{0}\} & |\mu_2| = {}^1\!/_2 \\ \rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \ldots & \end{array}$$

## Almost-Sure Termination (AST)

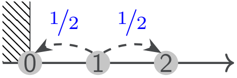
$|\mu_n|$ = probability mass on normal forms in $\mu_n$.

$\mathcal{P}$ is AST iff $\lim\limits_{n\to\infty} |\mu_n| = 1$ for every sequence $\mu_0 \rightrightarrows_{\mathcal{P}} \mu_1 \rightrightarrows_{\mathcal{P}} \cdots$

# Probabilistic Term Rewrite Systems (PTRS)

$\mathcal{P}_{\mathsf{rw}}$ $\quad \mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g}, \mathsf{g}),\ 1/2 : \mathsf{0}\}$

Right-hand side is a multi-distribution $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$.



*symmetric random walk*

$$
\begin{array}{lll}
 & \{1 : \mathsf{g}\} & |\mu_0| = 0 \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \{1/2 : \mathsf{c}(\mathsf{g}, \mathsf{g}), 1/2 : \mathsf{0}, \} & |\mu_1| = 1/2 \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \{1/4 : \mathsf{c}(\mathsf{c}(\mathsf{g}, \mathsf{g}), \mathsf{g}), 1/4 : \mathsf{c}(\mathsf{0}, \mathsf{g}), 1/2 : \mathsf{0}\} & |\mu_2| = 1/2 \\
\rightrightarrows_{\mathcal{P}_{\mathsf{rw}}} & \ldots & \xrightarrow{n\to\infty} 1
\end{array}
$$

## Almost-Sure Termination ($\mathtt{AST}$)

$|\mu_n|$ = probability mass on normal forms in $\mu_n$.

$\mathcal{P}$ is $\mathtt{AST}$ iff $\lim\limits_{n\to\infty} |\mu_n| = 1$ for every sequence $\mu_0 \rightrightarrows_{\mathcal{P}} \mu_1 \rightrightarrows_{\mathcal{P}} \cdots$

# Can We Decide Almost-Sure Termination?

- TRSs are Turing-complete $\implies$ termination and $\texttt{AST}$ are undecidable in general.

# Can We Decide Almost-Sure Termination?

- TRSs are Turing-complete $\implies$ termination and $\mathtt{AST}$ are undecidable in general.
- **Idea:** restrict the shape of the PTRS until $\mathtt{AST}$ becomes decidable.

# Can We Decide Almost-Sure Termination?

- TRSs are Turing-complete $\Longrightarrow$ termination and $\mathtt{AST}$ are undecidable in general.
- **Idea:** restrict the shape of the PTRS until $\mathtt{AST}$ becomes decidable.

## Roadmap of this talk

1. **Stochastic Context-Free Grammars (SCFGs)**: $\mathtt{AST}$ is decidable.

# Can We Decide Almost-Sure Termination?

- TRSs are Turing-complete $\implies$ termination and $\texttt{AST}$ are undecidable in general.
- **Idea:** restrict the shape of the PTRS until $\texttt{AST}$ becomes decidable.

## Roadmap of this talk

1. **Stochastic Context-Free Grammars (SCFGs)**: $\texttt{AST}$ is decidable.
2. **Ground**, **Non-Overlapping**, and **Tail-Recursive** PTRSs (class GNT)

# Can We Decide Almost-Sure Termination?

- TRSs are Turing-complete $\Longrightarrow$ termination and $\texttt{AST}$ are undecidable in general.
- **Idea:** restrict the shape of the PTRS until $\texttt{AST}$ becomes decidable.

## Roadmap of this talk

1. **Stochastic Context-Free Grammars (SCFGs)**: $\texttt{AST}$ is decidable.
2. **Ground**, **Non-Overlapping**, and **Tail-Recursive** PTRSs (class GNT)
3. **Right-ground**, **Non-Overlapping**, and **Tail-Recursive** PTRSs (class RNT $\supseteq$ GNT)

# Can We Decide Almost-Sure Termination?

- TRSs are Turing-complete $\Longrightarrow$ termination and $\mathtt{AST}$ are undecidable in general.
- **Idea:** restrict the shape of the PTRS until $\mathtt{AST}$ becomes decidable.

## Roadmap of this talk

1. **Stochastic Context-Free Grammars (SCFGs)**: $\mathtt{AST}$ is decidable.
2. **Ground**, **Non-Overlapping**, and **Tail-Recursive** PTRSs (class GNT)
3. **Right-ground**, **Non-Overlapping**, and **Tail-Recursive** PTRSs (class RNT $\supseteq$ GNT)

**This Talk** $\Longrightarrow$ first decision procedure for $\mathtt{AST}$ of a non-trivial class of PTRSs.

# Some Initial Observations

# Some Initial Observations

$$\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g}, \mathsf{g}), 1/2 : \mathsf{0}\}$$

$$\text{no drift} = 1/2 \cdot 2 + 1/2 \cdot 0 = \mathbf{1}$$

AST ✓

# Some Initial Observations

$$\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g}, \mathsf{g}), 1/2 : \mathsf{0}\}$$

no drift $= 1/2 \cdot 2 + 1/2 \cdot 0 = \mathbf{1}$

AST ✓

$$\mathsf{g} \to \{2/3 : \mathsf{c}(\mathsf{g}, \mathsf{g}), 1/3 : \mathsf{0}\}$$

positive drift $= 2/3 \cdot 2 + 1/3 \cdot 0 = 4/3 > \mathbf{1}$

not AST ✗

# Some Initial Observations

$$\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g},\mathsf{g}), 1/2 : \mathsf{0}\}$$

no drift $= 1/2 \cdot 2 + 1/2 \cdot 0 = \mathbf{1}$

AST ✓

$$\mathsf{g} \to \{2/3 : \mathsf{c}(\mathsf{g},\mathsf{g}), 1/3 : \mathsf{0}\}$$

positive drift $= 2/3 \cdot 2 + 1/3 \cdot 0 = 4/3 > \mathbf{1}$

not AST ✗

Same loop $\mathsf{g} \to^{+} \mathsf{c}(\mathsf{g},\mathsf{g})$ $\Longrightarrow$ the probabilities impact AST.

# Some Initial Observations

$$\mathsf{g} \to \{{}^{1}/{}_{2} : \mathsf{c}(\mathsf{g},\mathsf{g}), {}^{1}/{}_{2} : \mathsf{0}\}$$

no drift $= {}^{1}/{}_{2} \cdot 2 + {}^{1}/{}_{2} \cdot 0 = \mathbf{1}$

AST ✓

$$\mathsf{g} \to \{{}^{2}/{}_{3} : \mathsf{c}(\mathsf{g},\mathsf{g}), {}^{1}/{}_{3} : \mathsf{0}\}$$

positive drift $= {}^{2}/{}_{3} \cdot 2 + {}^{1}/{}_{3} \cdot 0 = {}^{4}/{}_{3} > \mathbf{1}$

not AST ✗

Same loop $\mathsf{g} \to^{+} \mathsf{c}(\mathsf{g},\mathsf{g})$ $\Longrightarrow$ the probabilities impact AST.

$$\mathsf{g} \to \{{}^{1}/{}_{2} : \mathsf{c}(\mathsf{g},\mathsf{g}), {}^{1}/{}_{2} : \mathsf{0}\}$$

no drift $= {}^{1}/{}_{2} \cdot 2 + {}^{1}/{}_{2} \cdot 0 = \mathbf{1}$

AST ✓

# Some Initial Observations

$$\mathsf{g} \to \{{}^{1}/_{2} : \mathsf{c}(\mathsf{g},\mathsf{g}), {}^{1}/_{2} : \mathsf{0}\}$$

no drift $= {}^{1}/_{2} \cdot 2 + {}^{1}/_{2} \cdot 0 = \mathbf{1}$

AST ✓

$$\mathsf{g} \to \{{}^{2}/_{3} : \mathsf{c}(\mathsf{g},\mathsf{g}), {}^{1}/_{3} : \mathsf{0}\}$$

positive drift $= {}^{2}/_{3} \cdot 2 + {}^{1}/_{3} \cdot 0 = {}^{4}/_{3} > \mathbf{1}$

not AST ✗

Same loop $\mathsf{g} \to^{+} \mathsf{c}(\mathsf{g},\mathsf{g})$ $\Longrightarrow$ the probabilities impact AST.

$$\mathsf{g} \to \{{}^{1}/_{2} : \mathsf{c}(\mathsf{g},\mathsf{g}), {}^{1}/_{2} : \mathsf{0}\}$$

no drift $= {}^{1}/_{2} \cdot 2 + {}^{1}/_{2} \cdot 0 = \mathbf{1}$

AST ✓

$$\mathsf{g} \to \{{}^{1}/_{2} : \mathsf{c}(\mathsf{g},\mathsf{g},\mathsf{g}), {}^{1}/_{2} : \mathsf{0}\}$$

positive drift $= {}^{1}/_{2} \cdot 3 + {}^{1}/_{2} \cdot 0 = {}^{3}/_{2} > \mathbf{1}$

not AST ✗

# Some Initial Observations

$\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g},\mathsf{g}), 1/2 : \mathsf{0}\}$

no drift $= 1/2 \cdot 2 + 1/2 \cdot 0 = \mathbf{1}$

AST ✓

$\mathsf{g} \to \{2/3 : \mathsf{c}(\mathsf{g},\mathsf{g}), 1/3 : \mathsf{0}\}$

positive drift $= 2/3 \cdot 2 + 1/3 \cdot 0 = 4/3 > \mathbf{1}$

not AST ✗

Same loop $\mathsf{g} \to^{+} \mathsf{c}(\mathsf{g},\mathsf{g})$ $\Longrightarrow$ the probabilities impact AST.

$\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g},\mathsf{g}), 1/2 : \mathsf{0}\}$

no drift $= 1/2 \cdot 2 + 1/2 \cdot 0 = \mathbf{1}$

AST ✓

$\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g},\mathsf{g},\mathsf{g}), 1/2 : \mathsf{0}\}$

positive drift $= 1/2 \cdot 3 + 1/2 \cdot 0 = 3/2 > \mathbf{1}$

not AST ✗

$\Longrightarrow$ the number of g's impact AST.

# Some Initial Observations

$\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g},\mathsf{g}), 1/2 : \mathsf{0}\}$

no drift $= 1/2 \cdot 2 + 1/2 \cdot 0 = \mathbf{1}$

AST ✓

$\mathsf{g} \to \{2/3 : \mathsf{c}(\mathsf{g},\mathsf{g}), 1/3 : \mathsf{0}\}$

positive drift $= 2/3 \cdot 2 + 1/3 \cdot 0 = 4/3 > \mathbf{1}$

not AST ✗

Same loop $\mathsf{g} \to^+ \mathsf{c}(\mathsf{g},\mathsf{g})$ $\implies$ the probabilities impact AST.

$\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g},\mathsf{g}), 1/2 : \mathsf{0}\}$

no drift $= 1/2 \cdot 2 + 1/2 \cdot 0 = \mathbf{1}$

AST ✓

$\mathsf{g} \to \{1/2 : \mathsf{c}(\mathsf{g},\mathsf{g},\mathsf{g}), 1/2 : \mathsf{0}\}$

positive drift $= 1/2 \cdot 3 + 1/2 \cdot 0 = 3/2 > \mathbf{1}$

not AST ✗

$\implies$ the number of g's impact AST.

We need a *stochastic model* that *counts recursive calls*: **SCFGs**.

# Stochastic Context-Free Grammars (SCFGs)

## SCFG

An *SCFG* $\mathcal{G} = (N, T, P)$ has:

- a set of non-terminals $N$
- a set of terminals $T$
- production rules $P : N \times (N \cup T)^* \to [0, 1]$ with $\sum_\pi P(A, \pi) = 1$ for each $A \in N$

# Stochastic Context-Free Grammars (SCFGs)

## SCFG

An *SCFG* $\mathcal{G} = (N, T, P)$ has:

- a set of non-terminals $N$
- a set of terminals $T$
- production rules $P : N \times (N \cup T)^* \to [0,1]$ with $\sum_\pi P(A,\pi) = 1$ for each $A \in N$

$$A \xrightarrow{2/3} AA \qquad A \xrightarrow{1/3} \varepsilon$$

$$N = \{A\}, \quad T = \varnothing$$

# Stochastic Context-Free Grammars (SCFGs)

## SCFG

An *SCFG* $\mathcal{G} = (N, T, P)$ has:

- a set of non-terminals $N$
- a set of terminals $T$
- production rules $P : N \times (N \cup T)^* \to [0,1]$ with $\sum_\pi P(A, \pi) = 1$ for each $A \in N$

$$A \xrightarrow{2/3} AA \qquad A \xrightarrow{1/3} \varepsilon$$

$$N = \{A\}, \quad T = \varnothing$$

Probabilistic choice replaces non-determinism.

# Stochastic Context-Free Grammars (SCFGs)

## SCFG

An *SCFG* $\mathcal{G} = (N, T, P)$ has:

- a set of non-terminals $N$
- a set of terminals $T$
- production rules $P : N \times (N \cup T)^* \to [0,1]$ with $\sum_\pi P(A, \pi) = 1$ for each $A \in N$

$$A \xrightarrow{2/3} AA \qquad A \xrightarrow{1/3} \varepsilon$$

$$N = \{A\}, \quad T = \varnothing$$

Probabilistic choice replaces non-determinism.

Reductions are **leftmost**:

$$1 : AA \;\Rightarrow_{\mathcal{G}}\; 2/3 : AAA$$

$$1 : AA \;\Rightarrow_{\mathcal{G}}\; 1/3 : \varepsilon A$$

# Stochastic Context-Free Grammars (SCFGs)

## SCFG

An *SCFG* $\mathcal{G} = (N, T, P)$ has:

- a set of non-terminals $N$
- a set of terminals $T$
- production rules $P : N \times (N \cup T)^* \to [0,1]$ with $\sum_\pi P(A, \pi) = 1$ for each $A \in N$

$$A \xrightarrow{2/3} AA \qquad A \xrightarrow{1/3} \varepsilon$$

$$N = \{A\}, \quad T = \varnothing$$

AST is decidable for SCFGs
[Etessami & Yannakakis'09]

Probabilistic choice replaces non-determinism.

Reductions are **leftmost**:

$$1 : AA \quad \begin{matrix} \Rightarrow_{\mathcal{G}} \\ \Rightarrow_{\mathcal{G}} \end{matrix} \quad \begin{matrix} 2/3 : AAA \\ 1/3 : \varepsilon A \end{matrix}$$

# Step 1: Productivity

## Productive SCFG

$\mathcal{G}$ is *productive* if every non-terminal $A \in N$ can reach a terminal word, i.e. $1 : A \Rightarrow_{\mathcal{G}}^{*} p : \alpha$ with $\alpha \in T^{*}$.

# Step 1: Productivity

## Productive SCFG

$\mathcal{G}$ is *productive* if every non-terminal $A \in N$ can reach a terminal word, i.e. $1 : A \Rightarrow_{\mathcal{G}}^{*} p : \alpha$ with $\alpha \in T^*$.

$$A \xrightarrow{1/2} AA \qquad A \xrightarrow{1/2} \varepsilon$$

$$1 : A \Rightarrow_{\mathcal{G}} 1/2 : \varepsilon \in T^*$$

$\Rightarrow$ productive ✓

# Step 1: Productivity

## Productive SCFG

$\mathcal{G}$ is *productive* if every non-terminal $A \in N$ can reach a terminal word, i.e. $1 : A \Rightarrow_{\mathcal{G}}^{*} p : \alpha$ with $\alpha \in T^{*}$.

$$A \xrightarrow{1/2} AA \qquad A \xrightarrow{1/2} \varepsilon$$

$$1 : A \Rightarrow_{\mathcal{G}} 1/2 : \varepsilon \in T^{*}$$

$\Rightarrow$ productive ✓

$$B \xrightarrow{1} BB$$

$$1 : B \Rightarrow_{\mathcal{G}} 1 : BB \Rightarrow_{\mathcal{G}} 1 : BBB \Rightarrow_{\mathcal{G}} \cdots$$

$\Rightarrow$ not productive ✗ $\Rightarrow$ not AST

# Step 1: Productivity

## Productive SCFG

$\mathcal{G}$ is *productive* if every non-terminal $A \in N$ can reach a terminal word, i.e. $1 : A \Rightarrow_{\mathcal{G}}^{*} p : \alpha$ with $\alpha \in T^*$.

$$A \xrightarrow{1/2} AA \qquad A \xrightarrow{1/2} \varepsilon$$

$$1 : A \Rightarrow_{\mathcal{G}} 1/2 : \varepsilon \in T^*$$

$\Rightarrow$ productive ✓

$$B \xrightarrow{1} BB$$

$$1 : B \Rightarrow_{\mathcal{G}} 1 : BB \Rightarrow_{\mathcal{G}} 1 : BBB \Rightarrow_{\mathcal{G}} \cdots$$

$\Rightarrow$ not productive ✗ $\Rightarrow$ not AST

But productivity is **not enough**: the productive $A \xrightarrow{2/3} AA$, $A \xrightarrow{1/3} \varepsilon$ reaches $\varepsilon$ (but is not AST).

# Step 1: Productivity

## Productive SCFG

$\mathcal{G}$ is *productive* if every non-terminal $A \in N$ can reach a terminal word, i.e. $1 : A \Rightarrow_{\mathcal{G}}^{*} p : \alpha$ with $\alpha \in T^*$.

$$A \xrightarrow{1/2} AA \qquad A \xrightarrow{1/2} \varepsilon$$

$$1 : A \Rightarrow_{\mathcal{G}} 1/2 : \varepsilon \in T^*$$

$\Rightarrow$ productive ✓

$$B \xrightarrow{1} BB$$

$$1 : B \Rightarrow_{\mathcal{G}} 1 : BB \Rightarrow_{\mathcal{G}} 1 : BBB \Rightarrow_{\mathcal{G}} \cdots$$

$\Rightarrow$ not productive ✗ $\Rightarrow$ not AST

But productivity is **not enough**: the productive $A \xrightarrow{2/3} AA$, $A \xrightarrow{1/3} \varepsilon$ reaches $\varepsilon$ (but is not AST).

We must **count** how many non-terminals each step creates.

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \ldots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} \;=\; \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \ldots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} \; = \sum_{S_i \xrightarrow{p} \alpha \in P} {\color{blue}p} \cdot |\alpha|_{S_j}$$

$$\mathsf{A} \xrightarrow{{\color{blue}1/3}} \mathsf{A} \qquad \mathsf{A} \xrightarrow{{\color{blue}1/3}} \mathsf{BA} \qquad \mathsf{A} \xrightarrow{{\color{blue}1/3}} \varepsilon$$

$$\mathsf{B} \xrightarrow{{\color{blue}1/2}} \mathsf{A} \qquad \mathsf{B} \xrightarrow{{\color{blue}1/2}} \varepsilon$$

$$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$$

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \ldots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} \;=\; \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

$$B_{\mathcal{G}} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$\mathsf{A} \xrightarrow{1/3} \mathsf{A} \qquad \mathsf{A} \xrightarrow{1/3} \mathsf{BA} \qquad \mathsf{A} \xrightarrow{1/3} \varepsilon$

$\mathsf{B} \xrightarrow{1/2} \mathsf{A} \qquad \mathsf{B} \xrightarrow{1/2} \varepsilon$

$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \ldots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} \;=\; \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

$\mathsf{A} \xrightarrow{1/3} \mathsf{A}$ $\quad$ $\mathsf{A} \xrightarrow{1/3} \mathsf{BA}$ $\quad$ $\mathsf{A} \xrightarrow{1/3} \varepsilon$

$\mathsf{B} \xrightarrow{1/2} \mathsf{A}$ $\quad$ $\mathsf{B} \xrightarrow{1/2} \varepsilon$

$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$

$$B_{\mathcal{G}} = \begin{pmatrix} 2/3 & \\ & \end{pmatrix}$$

$(B_{\mathcal{G}})_{1,1}$: one A in two A-rules $\Rightarrow 1/3 \cdot 1 + 1/3 \cdot 1 = 2/3$

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \ldots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} \;=\; \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

$$\mathsf{A} \xrightarrow{1/3} \mathsf{A} \qquad \mathsf{A} \xrightarrow{1/3} \mathsf{BA} \qquad \mathsf{A} \xrightarrow{1/3} \varepsilon$$

$$\mathsf{B} \xrightarrow{1/2} \mathsf{A} \qquad \mathsf{B} \xrightarrow{1/2} \varepsilon$$

$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$

$$B_{\mathcal{G}} = \begin{pmatrix} 2/3 & 1/3 \\ & \end{pmatrix}$$

$(B_{\mathcal{G}})_{1,2}$: one B in A's rules $\Rightarrow 1/3 \cdot 1 = 1/3$

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \ldots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} = \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

$$\mathsf{A} \xrightarrow{1/3} \mathsf{A} \qquad \mathsf{A} \xrightarrow{1/3} \mathsf{BA} \qquad \mathsf{A} \xrightarrow{1/3} \varepsilon$$

$$\mathsf{B} \xrightarrow{1/2} \mathsf{A} \qquad \mathsf{B} \xrightarrow{1/2} \varepsilon$$

$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$

$$B_{\mathcal{G}} = \begin{pmatrix} 2/3 & 1/3 \\ 1/2 & \end{pmatrix}$$

$(B_{\mathcal{G}})_{2,1}$: one A in B's rules $\Rightarrow 1/2 \cdot 1 = 1/2$

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \dots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} = \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

$$\mathsf{A} \xrightarrow{1/3} \mathsf{A} \qquad \mathsf{A} \xrightarrow{1/3} \mathsf{BA} \qquad \mathsf{A} \xrightarrow{1/3} \varepsilon$$

$$\mathsf{B} \xrightarrow{1/2} \mathsf{A} \qquad \mathsf{B} \xrightarrow{1/2} \varepsilon$$

$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$

$$B_{\mathcal{G}} = \begin{pmatrix} 2/3 & 1/3 \\ 1/2 & 0 \end{pmatrix}$$

$(B_{\mathcal{G}})_{2,2}$: no B in B's rules $\Rightarrow 0$

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \ldots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} \; = \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

$$\mathsf{A} \xrightarrow{1/3} \mathsf{A} \qquad \mathsf{A} \xrightarrow{1/3} \mathsf{BA} \qquad \mathsf{A} \xrightarrow{1/3} \varepsilon$$

$$\mathsf{B} \xrightarrow{1/2} \mathsf{A} \qquad \mathsf{B} \xrightarrow{1/2} \varepsilon$$

$$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$$

$$B_{\mathcal{G}} = \begin{pmatrix} 2/3 & 1/3 \\ 1/2 & 0 \end{pmatrix}$$

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \ldots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} \;=\; \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

$$\mathsf{A} \xrightarrow{1/3} \mathsf{A} \qquad \mathsf{A} \xrightarrow{1/3} \mathsf{BA} \qquad \mathsf{A} \xrightarrow{1/3} \varepsilon$$

$$\mathsf{B} \xrightarrow{1/2} \mathsf{A} \qquad \mathsf{B} \xrightarrow{1/2} \varepsilon$$

$$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$$

$$B_{\mathcal{G}} = \begin{pmatrix} 2/3 & 1/3 \\ 1/2 & 0 \end{pmatrix}$$

**Theorem: AST via the Spectral Radius [Etessami & Yannakakis'09]**

A productive SCFG $\mathcal{G}$ is AST $\iff$ the spectral radius $\rho(B_{\mathcal{G}}) \leq 1$.

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_\mathcal{G}$: entry $(B_\mathcal{G})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \ldots, S_k\}\big)$

$$(B_\mathcal{G})_{i,j} = \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

$$\mathsf{A} \xrightarrow{1/3} \mathsf{A} \qquad \mathsf{A} \xrightarrow{1/3} \mathsf{BA} \qquad \mathsf{A} \xrightarrow{1/3} \varepsilon$$

$$\mathsf{B} \xrightarrow{1/2} \mathsf{A} \qquad \mathsf{B} \xrightarrow{1/2} \varepsilon$$

$$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$$

$$B_\mathcal{G} = \begin{pmatrix} 2/3 & 1/3 \\ 1/2 & 0 \end{pmatrix}$$

**Theorem: AST via the Spectral Radius [Etessami & Yannakakis'09]**

A productive SCFG $\mathcal{G}$ is AST $\iff$ the spectral radius $\rho(B_\mathcal{G}) \leq 1$.

$\rho(B_\mathcal{G})$ = largest absolute value among the eigenvalues (roots of the characteristic polynomial $\chi_{B_\mathcal{G}}$).

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \dots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} = \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

$\mathsf{A} \xrightarrow{1/3} \mathsf{A}$ $\quad$ $\mathsf{A} \xrightarrow{1/3} \mathsf{BA}$ $\quad$ $\mathsf{A} \xrightarrow{1/3} \varepsilon$

$\mathsf{B} \xrightarrow{1/2} \mathsf{A}$ $\quad$ $\mathsf{B} \xrightarrow{1/2} \varepsilon$

$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$

$$B_{\mathcal{G}} = \begin{pmatrix} 2/3 & 1/3 \\ 1/2 & 0 \end{pmatrix}$$

$$\chi_{B_{\mathcal{G}}} = X^2 - 2/3X - 1/6$$

**Theorem: AST via the Spectral Radius [Etessami & Yannakakis'09]**

A productive SCFG $\mathcal{G}$ is AST $\iff$ the spectral radius $\rho(B_{\mathcal{G}}) \leq 1$.

$\rho(B_{\mathcal{G}})$ = largest absolute value among the eigenvalues (roots of the characteristic polynomial $\chi_{B_{\mathcal{G}}}$).

# Step 2: Counting via the Moment Matrix

**Moment matrix** $B_{\mathcal{G}}$: entry $(B_{\mathcal{G}})_{i,j}$ = expected number of $S_j$ created from $S_i$ $\big(N = \{S_1, \ldots, S_k\}\big)$

$$(B_{\mathcal{G}})_{i,j} \;=\; \sum_{S_i \xrightarrow{p} \alpha \in P} p \cdot |\alpha|_{S_j}$$

$$\mathsf{A} \xrightarrow{1/3} \mathsf{A} \qquad \mathsf{A} \xrightarrow{1/3} \mathsf{BA} \qquad \mathsf{A} \xrightarrow{1/3} \varepsilon$$

$$\mathsf{B} \xrightarrow{1/2} \mathsf{A} \qquad \mathsf{B} \xrightarrow{1/2} \varepsilon$$

$$S_1 = \mathsf{A}, \quad S_2 = \mathsf{B}$$

$$B_{\mathcal{G}} = \begin{pmatrix} 2/3 & 1/3 \\ 1/2 & 0 \end{pmatrix}$$

$$\chi_{B_{\mathcal{G}}} = X^2 - 2/3X - 1/6$$

$$\rho(B_{\mathcal{G}}) = \tfrac{2+\sqrt{10}}{6} \approx 0.86 \leq 1 \Rightarrow \mathcal{G} \text{ is } \mathtt{AST}$$

**Theorem: AST via the Spectral Radius [Etessami & Yannakakis'09]**

A productive SCFG $\mathcal{G}$ is $\mathtt{AST}$ $\iff$ the spectral radius $\rho(B_{\mathcal{G}}) \leq 1$.

$\rho(B_{\mathcal{G}})$ = largest absolute value among the eigenvalues (roots of the characteristic polynomial $\chi_{B_{\mathcal{G}}}$).

# Which PTRSs Become SCFGs? The Class GNT

Three restrictions turn a PTRS into a grammar:

# Which PTRSs Become SCFGs? The Class GNT

Three restrictions turn a PTRS into a grammar:

## Ground — no variables in the rules

✓ $\mathsf{g} \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{g})\}$ ✗ $\mathsf{f}(x) \to \{1 : \mathsf{g}(x)\}$ ✗ $\mathsf{f}(x) \to \{1 : \mathsf{f}(\mathsf{a})\}$

# Which PTRSs Become SCFGs? The Class GNT

Three restrictions turn a PTRS into a grammar:

**G**round — no variables in the rules

✓ $\mathsf{g} \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{g})\}$ ✗ $\mathsf{f}(x) \to \{1 : \mathsf{g}(x)\}$ ✗ $\mathsf{f}(x) \to \{1 : \mathsf{f}(\mathsf{a})\}$

**N**on-**O**verlapping — no two left-hand sides unify

✓ $\mathsf{a} \to \ldots, \quad \mathsf{f}(\mathsf{b}) \to \ldots$ ✗ $\mathsf{a} \to \ldots, \quad \mathsf{f}(\mathsf{a}) \to \ldots$

# Which PTRSs Become SCFGs? The Class GNT

Three restrictions turn a PTRS into a grammar:

**G**round — no variables in the rules

✓ $\mathsf{g} \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{g})\}$ ✗ $\mathsf{f}(x) \to \{1 : \mathsf{g}(x)\}$ ✗ $\mathsf{f}(x) \to \{1 : \mathsf{f}(\mathsf{a})\}$

**N**on-**O**verlapping — no two left-hand sides unify

✓ $\mathsf{a} \to \ldots, \quad \mathsf{f}(\mathsf{b}) \to \ldots$ ✗ $\mathsf{a} \to \ldots, \quad \mathsf{f}(\mathsf{a}) \to \ldots$

**T**ail-**R**ecursive — no defined symbol below another

✓ $\mathsf{g} \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{g})\}$ ✗ $\mathsf{f}(x) \to \{1 : \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{a})))\}$

# Which PTRSs Become SCFGs? The Class GNT

Three restrictions turn a PTRS into a grammar:

**G**round — no variables in the rules

✓ $\mathsf{g} \to \{1 : \mathsf{c}(\mathsf{g},\mathsf{g})\}$ ✗ $\mathsf{f}(x) \to \{1 : \mathsf{g}(x)\}$ ✗ $\mathsf{f}(x) \to \{1 : \mathsf{f}(\mathsf{a})\}$

**N**on-**O**verlapping — no two left-hand sides unify

✓ $\mathsf{a} \to \ldots,\quad \mathsf{f}(\mathsf{b}) \to \ldots$ ✗ $\mathsf{a} \to \ldots,\quad \mathsf{f}(\mathsf{a}) \to \ldots$

**T**ail-**R**ecursive — no defined symbol below another

✓ $\mathsf{g} \to \{1 : \mathsf{c}(\mathsf{g},\mathsf{g})\}$ ✗ $\mathsf{f}(x) \to \{1 : \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{a})))\}$

GNT

GNT $=$ **G**round $+$ **N**on-**O**verlapping $+$ **T**ail-**R**ecursive

# Extracting the Grammar $\mathcal{T}_{\text{gram}}(\mathcal{P})$

Keep only non-terminals $N = lhs(\mathcal{P}) = \{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}), \mathsf{g}\}$, $\quad T = \varnothing$.

$$\mathsf{f}(\mathsf{a}) \to \{{}^1\!/\!{}_3 : \mathsf{c}(\mathsf{g}, \mathsf{c}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b}))),\ \ {}^2\!/\!{}_3 : \mathsf{c}(\mathsf{a}, \mathsf{b})\}$$

$$\mathsf{f}(\mathsf{b}) \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{b})\}$$

$$\mathsf{g} \to \{1 : \mathsf{f}(\mathsf{a})\}$$

# Extracting the Grammar $\mathcal{T}_{\text{gram}}(\mathcal{P})$

Keep only non-terminals $N = lhs(\mathcal{P}) = \{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}), \mathsf{g}\}$, $\quad T = \varnothing$.

$$\mathsf{f}(\mathsf{a}) \to \{{}^1/_3 : \mathsf{c}(\mathsf{g}, \mathsf{c}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))),\ \ {}^2/_3 : \mathsf{c}(\mathsf{a}, \mathsf{b})\}$$

$$\mathsf{f}(\mathsf{b}) \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{b})\}$$

$$\mathsf{g} \to \{1 : \mathsf{f}(\mathsf{a})\}$$

# Extracting the Grammar $\mathcal{T}_{\mathrm{gram}}(\mathcal{P})$

Keep only non-terminals $N = lhs(\mathcal{P}) = \{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}), \mathsf{g}\}, \quad T = \varnothing$.

$$\mathsf{f}(\mathsf{a}) \to \{{}^1\!/_3 : \mathsf{c}(\mathsf{g}, \mathsf{c}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b}))),\ \ {}^2\!/_3 : \mathsf{c}(\mathsf{a}, \mathsf{b})\} \quad \rightsquigarrow \quad \begin{array}{l} \mathsf{f}(\mathsf{a}) \xrightarrow{{}^1\!/_3} \mathsf{g}\,\mathsf{f}(\mathsf{a})\,\mathsf{f}(\mathsf{b}) \\ \mathsf{f}(\mathsf{a}) \xrightarrow{{}^2\!/_3} \varepsilon \end{array}$$

$$\mathsf{f}(\mathsf{b}) \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{b})\}$$

$$\mathsf{g} \to \{1 : \mathsf{f}(\mathsf{a})\}$$

# Extracting the Grammar $\mathcal{T}_{\mathrm{gram}}(\mathcal{P})$

Keep only non-terminals $N = lhs(\mathcal{P}) = \{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}), \mathsf{g}\}, \quad T = \varnothing$.

$$\mathsf{f}(\mathsf{a}) \to \{{}^{1}/{}_{3} : \mathsf{c}(\mathsf{g}, \mathsf{c}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b}))),\ {}^{2}/{}_{3} : \mathsf{c}(\mathsf{a}, \mathsf{b})\} \quad \rightsquigarrow \quad \begin{array}{l} \mathsf{f}(\mathsf{a}) \xrightarrow{1/3} \mathsf{g}\,\mathsf{f}(\mathsf{a})\,\mathsf{f}(\mathsf{b}) \\ \mathsf{f}(\mathsf{a}) \xrightarrow{2/3} \varepsilon \end{array}$$

$$\mathsf{f}(\mathsf{b}) \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{b})\}$$

$$\mathsf{g} \to \{1 : \mathsf{f}(\mathsf{a})\}$$

# Extracting the Grammar $\mathcal{T}_{\text{gram}}(\mathcal{P})$

Keep only non-terminals $N = \mathit{lhs}(\mathcal{P}) = \{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}), \mathsf{g}\}$, $\quad T = \varnothing$.

$$\mathsf{f}(\mathsf{a}) \to \{{}^1\!/_3 : \mathsf{c}(\mathsf{g}, \mathsf{c}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b}))),\ \ {}^2\!/_3 : \mathsf{c}(\mathsf{a}, \mathsf{b})\} \quad \rightsquigarrow \quad \begin{array}{l} \mathsf{f}(\mathsf{a}) \xrightarrow{1/3} \mathsf{g}\,\mathsf{f}(\mathsf{a})\,\mathsf{f}(\mathsf{b}) \\ \mathsf{f}(\mathsf{a}) \xrightarrow{2/3} \varepsilon \end{array}$$

$$\mathsf{f}(\mathsf{b}) \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{b})\} \quad \rightsquigarrow \quad \mathsf{f}(\mathsf{b}) \xrightarrow{1} \mathsf{g}$$

$$\mathsf{g} \to \{1 : \mathsf{f}(\mathsf{a})\}$$

# Extracting the Grammar $\mathcal{T}_{\mathrm{gram}}(\mathcal{P})$

Keep only non-terminals $N = \mathit{lhs}(\mathcal{P}) = \{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}), \mathsf{g}\}, \quad T = \varnothing$.

$$\mathsf{f}(\mathsf{a}) \to \{{}^{1}/{}_{3} : \mathsf{c}(\mathsf{g}, \mathsf{c}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))),\ \ {}^{2}/{}_{3} : \mathsf{c}(\mathsf{a}, \mathsf{b})\} \quad \leadsto \quad \begin{array}{l} \mathsf{f}(\mathsf{a}) \xrightarrow{1/3} \mathsf{g}\,\mathsf{f}(\mathsf{a})\,\mathsf{f}(\mathsf{b}) \\ \mathsf{f}(\mathsf{a}) \xrightarrow{2/3} \varepsilon \end{array}$$

$$\mathsf{f}(\mathsf{b}) \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{b})\} \quad \leadsto \quad \mathsf{f}(\mathsf{b}) \xrightarrow{1} \mathsf{g}$$

$$\mathsf{g} \to \{1 : \mathsf{f}(\mathsf{a})\}$$

# Extracting the Grammar $\mathcal{T}_{\mathrm{gram}}(\mathcal{P})$

Keep only non-terminals $N = lhs(\mathcal{P}) = \{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}), \mathsf{g}\}, \quad T = \varnothing$.

$$\mathsf{f}(\mathsf{a}) \to \{{}^1\!/\!{}_3 : \mathsf{c}(\mathsf{g}, \mathsf{c}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b}))),\ \ {}^2\!/\!{}_3 : \mathsf{c}(\mathsf{a}, \mathsf{b})\} \quad \rightsquigarrow \quad \begin{array}{l} \mathsf{f}(\mathsf{a}) \xrightarrow{1/3} \mathsf{g}\,\mathsf{f}(\mathsf{a})\,\mathsf{f}(\mathsf{b}) \\ \mathsf{f}(\mathsf{a}) \xrightarrow{2/3} \varepsilon \end{array}$$

$$\mathsf{f}(\mathsf{b}) \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{b})\} \quad \rightsquigarrow \quad \mathsf{f}(\mathsf{b}) \xrightarrow{1} \mathsf{g}$$

$$\mathsf{g} \to \{1 : \mathsf{f}(\mathsf{a})\} \quad \rightsquigarrow \quad \mathsf{g} \xrightarrow{1} \mathsf{f}(\mathsf{a})$$

# Extracting the Grammar $\mathcal{T}_{\mathrm{gram}}(\mathcal{P})$

Keep only non-terminals $N = \mathit{lhs}(\mathcal{P}) = \{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}), \mathsf{g}\}$, $\quad T = \varnothing$.

$$\mathsf{f}(\mathsf{a}) \to \{{}^1\!/\!_3 : \mathsf{c}(\mathsf{g}, \mathsf{c}(\mathsf{f}(\mathsf{a}),\mathsf{f}(\mathsf{b}))),\ \ {}^2\!/\!_3 : \mathsf{c}(\mathsf{a}, \mathsf{b})\} \quad \rightsquigarrow \quad \begin{array}{l} \mathsf{f}(\mathsf{a}) \xrightarrow{1/3} \mathsf{g}\,\mathsf{f}(\mathsf{a})\,\mathsf{f}(\mathsf{b}) \\ \mathsf{f}(\mathsf{a}) \xrightarrow{2/3} \varepsilon \end{array}$$

$$\mathsf{f}(\mathsf{b}) \to \{1 : \mathsf{c}(\mathsf{g}, \mathsf{b})\} \quad \rightsquigarrow \quad \mathsf{f}(\mathsf{b}) \xrightarrow{1} \mathsf{g}$$

$$\mathsf{g} \to \{1 : \mathsf{f}(\mathsf{a})\} \quad \rightsquigarrow \quad \mathsf{g} \xrightarrow{1} \mathsf{f}(\mathsf{a})$$

## Lemma

For $\mathcal{P} \in$ GNT: $\quad \mathcal{P}$ is $\mathtt{AST} \iff \mathcal{T}_{\mathrm{gram}}(\mathcal{P})$ is $\mathtt{AST}$.

# The Transformation $\mathcal{T}_{\mathsf{GNT}}$ : RNT $\to$ GNT

## RNT

RNT $=$ all PTRSs that are **R**ight-ground, **N**on-**O**verlapping, and **T**ail-**R**ecursive. GNT $\subseteq$ RNT

**Idea:** create one ground copy of each rule per relevant instance:

$$\mathcal{T}_{\mathsf{GNT}}(\mathcal{P}) = \{\, t \to \mu \mid t \in \mathrm{instLhs}(\mathcal{P}),\ \ell \to \mu \in \mathcal{P},\ t = \ell\sigma \,\}.$$

# The Transformation $\mathcal{T}_{\mathsf{GNT}}$ : RNT → GNT

**RNT**

RNT = all PTRSs that are **R**ight-ground, **N**on-**O**verlapping, and **T**ail-**R**ecursive. GNT ⊆ RNT

**Idea:** create one ground copy of each rule per relevant instance:

$$\mathcal{T}_{\mathsf{GNT}}(\mathcal{P}) = \{ t \to \mu \mid t \in \mathrm{instLhs}(\mathcal{P}),\ \ell \to \mu \in \mathcal{P},\ t = \ell\sigma \}.$$

$\mathcal{P} \in$ RNT

$$\mathsf{f}(x) \to \{{}^1\!/_2 : \mathsf{g}, {}^1\!/_2 : \mathsf{c}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))\}$$

# The Transformation $\mathcal{T}_{\mathsf{GNT}} : \mathsf{RNT} \to \mathsf{GNT}$

**RNT**

RNT $=$ all PTRSs that are **R**ight-ground, **N**on-**O**verlapping, and **T**ail-**R**ecursive. GNT $\subseteq$ RNT

**Idea:** create one ground copy of each rule per relevant instance:

$$\mathcal{T}_{\mathsf{GNT}}(\mathcal{P}) = \{\, t \to \mu \mid t \in \mathrm{instLhs}(\mathcal{P}),\ \ell \to \mu \in \mathcal{P},\ t = \ell\sigma \,\}.$$

$\mathcal{P} \in \mathsf{RNT}$

$$\mathsf{f}(x) \to \{1/2 : \mathsf{g}, 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))\}$$

$\xRightarrow[\text{instantiate } x \in \{\mathsf{a},\mathsf{b}\}]{\mathcal{T}_{\mathsf{GNT}}}$

$\mathcal{T}_{\mathsf{GNT}}(\mathcal{P}) \in \mathsf{GNT}$

$$\mathsf{f}(\mathsf{a}) \to \{1/2 : \mathsf{g}, 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))\}$$
$$\mathsf{f}(\mathsf{b}) \to \{1/2 : \mathsf{g}, 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))\}$$

# The Transformation $\mathcal{T}_{\mathsf{GNT}} : \mathsf{RNT} \to \mathsf{GNT}$

## RNT

RNT $=$ all PTRSs that are **R**ight-ground, **N**on-**O**verlapping, and **T**ail-**R**ecursive. GNT $\subseteq$ RNT

**Idea:** create one ground copy of each rule per relevant instance:

$$\mathcal{T}_{\mathsf{GNT}}(\mathcal{P}) = \{\, t \to \mu \mid t \in \mathrm{instLhs}(\mathcal{P}),\ \ell \to \mu \in \mathcal{P},\ t = \ell\sigma \,\}.$$

$\mathcal{P} \in$ RNT

$$\mathsf{f}(x) \to \{1/2 : \mathsf{g},\ 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))\}$$

$$\xRightarrow[\text{instantiate } x \in \{\mathsf{a}, \mathsf{b}\}]{\mathcal{T}_{\mathsf{GNT}}}$$

$\mathcal{T}_{\mathsf{GNT}}(\mathcal{P}) \in$ GNT

$$\mathsf{f}(\mathsf{a}) \to \{1/2 : \mathsf{g},\ 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))\}$$
$$\mathsf{f}(\mathsf{b}) \to \{1/2 : \mathsf{g},\ 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b}))\}$$

## Lemma

For $\mathcal{P} \in$ RNT: $\mathcal{T}_{\mathsf{GNT}}(\mathcal{P}) \in$ GNT and $\mathcal{P}$ is $\mathtt{AST}$ $\iff$ $\mathcal{T}_{\mathsf{GNT}}(\mathcal{P})$ is $\mathtt{AST}$.

# The Complete Decision Procedure for RNT

**Theorem**

$\texttt{AST}$ is decidable for $\mathcal{P} \in \mathsf{RNT}$

$$\begin{tikzcd} \boxed{\mathcal{P} \in \mathsf{RNT}} \arrow[r, "\text{instantiate}"] & \boxed{\mathcal{T}_{\mathsf{GNT}}(\mathcal{P}) \in \mathsf{GNT}} \arrow[r, "\text{extract}"] & \boxed{\mathcal{T}_{\mathrm{gram}}(\mathcal{P}) \in \mathsf{SCFG}} \arrow[r, "\text{productive?}"] & \boxed{\rho(B_{\mathcal{G}}) \leq 1?} \end{tikzcd}$$

# Evaluation

Implemented in AProVE, run on the $173$ probabilistic TRSs of the *Termination Problem Data Base*.

# Evaluation

Implemented in AProVE, run on the $173$ probabilistic TRSs of the *Termination Problem Data Base*.

| | GNT | RNT |
|---|---|---|
| PTRSs in the class | $24$ | $32$ |
| `AST` proven | $18$ | $23$ |
| `AST` disproven | $5$ | $8$ |

# Evaluation

Implemented in AProVE, run on the $173$ probabilistic TRSs of the *Termination Problem Data Base*.

| | GNT | RNT |
|---|---|---|
| PTRSs in the class | $24$ | $32$ |
| AST proven | $18$ | $23$ |
| AST disproven | $5$ | $8$ |

- a *decision* procedure: decides AST for *every* $\mathcal{P} \in$ RNT ...

# Evaluation

Implemented in AProVE, run on the $173$ probabilistic TRSs of the *Termination Problem Data Base*.

| | GNT | RNT |
|---|---|---|
| PTRSs in the class | $24$ | $32$ |
| `AST` proven | $18$ | $23$ |
| `AST` disproven | $5$ | $8$ |

- a *decision* procedure: decides `AST` for *every* $\mathcal{P} \in$ RNT ...

  ... all but one, whose SMT query exceeds the timeout ...

# Evaluation

Implemented in AProVE, run on the $173$ probabilistic TRSs of the *Termination Problem Data Base*.

| | GNT | RNT |
|---|---|---|
| PTRSs in the class | $24$ | $32$ |
| AST proven | $18$ | $23$ |
| AST disproven | $5$ | $8$ |

- a *decision* procedure: decides AST for *every* $\mathcal{P} \in$ RNT ...

  ... all but one, whose SMT query exceeds the timeout ...

  ... that one is still proven AST by ranking functions

# Conclusion

## Contributions

- First decision procedure for AST of a non-trivial class of PTRSs.
- **Ground** (GNT): transform to an AST-equivalent **SCFG**, decide via the *spectral radius*.
- **Right-ground** (RNT): instantiate to GNT, preserving AST.

# Conclusion

## Contributions

- First decision procedure for $\mathtt{AST}$ of a non-trivial class of PTRSs.
- **Ground** (GNT): transform to an $\mathtt{AST}$-equivalent **SCFG**, decide via the *spectral radius*.
- **Right-ground** (RNT): instantiate to GNT, preserving $\mathtt{AST}$.

$$\text{RNT} \xrightarrow{\text{instantiate}} \text{GNT} \xrightarrow{\text{extract}} \text{SCFG} \xrightarrow{\text{spectral radius}} \rho(B_{\mathcal{G}}) \leq 1$$

# Conclusion

## Contributions

- First decision procedure for $\mathtt{AST}$ of a non-trivial class of PTRSs.
- **Ground** (GNT): transform to an $\mathtt{AST}$-equivalent **SCFG**, decide via the *spectral radius*.
- **Right-ground** (RNT): instantiate to GNT, preserving $\mathtt{AST}$.

$$\text{RNT} \xrightarrow{\text{instantiate}} \text{GNT} \xrightarrow{\text{extract}} \text{SCFG} \xrightarrow{\text{spectral radius}} \rho(B_{\mathcal{G}}) \leq 1$$

## Open Problems

- Can we drop *non-overlappingness* (deal with non-determinism)?
- Can we drop *tail-recursiveness* (deal with unbounded memory)?
- Decision procedures for *PAST* and expected complexity?

# Why Term Rewriting in Program Analysis?

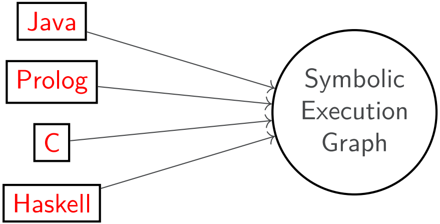
$\boxed{\text{Java}}$

$\boxed{\text{Prolog}}$

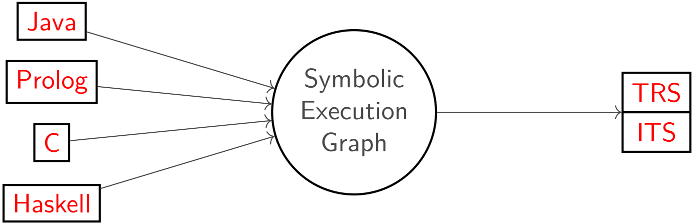
$\boxed{\text{C}}$

$\boxed{\text{Haskell}}$

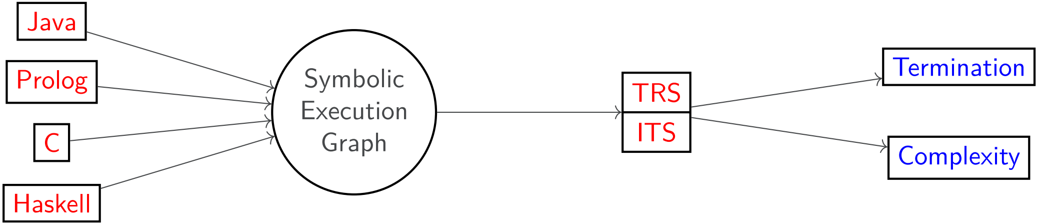
# Why Term Rewriting in Program Analysis?



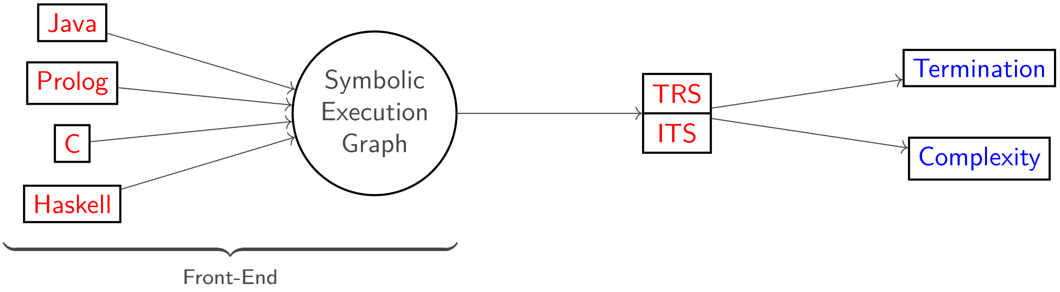
# Why Term Rewriting in Program Analysis?



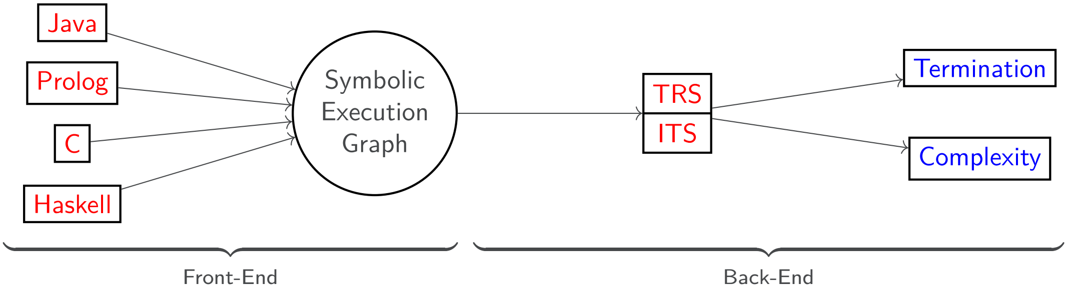
# Why Term Rewriting in Program Analysis?



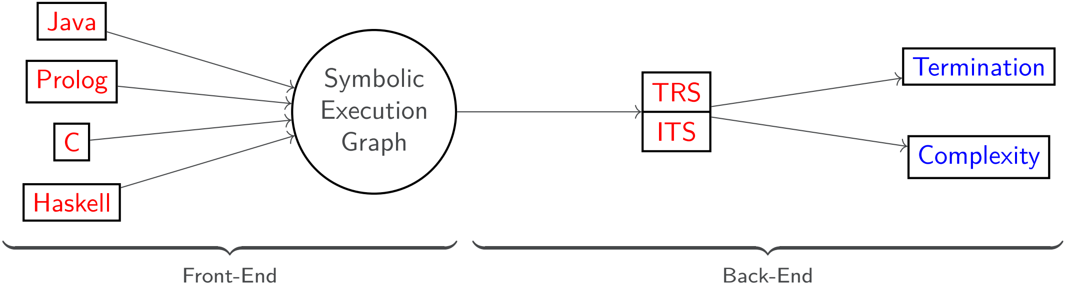
# Why Term Rewriting in Program Analysis?



# Why Term Rewriting in Program Analysis?

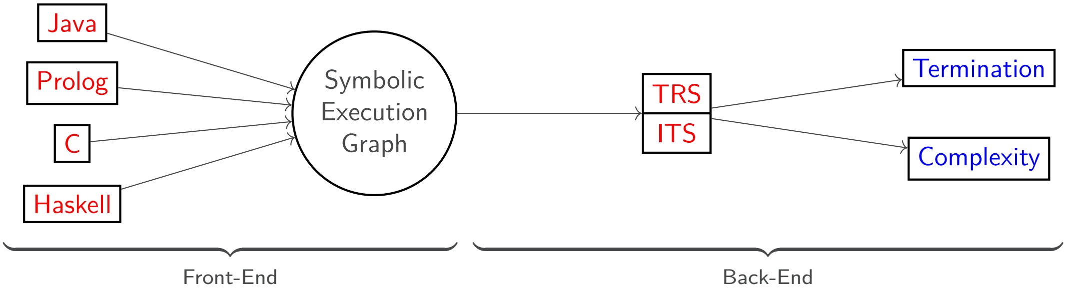


# Why Term Rewriting in Program Analysis?




- **Term Rewrite Systems (TRSs)** in the back-end for automated program analysis (used in AProVE, at the *Termination Competition* since 2004)

# Why Term Rewriting in Program Analysis?




- **Term Rewrite Systems (TRSs)** in the back-end for automated program analysis (used in AProVE, at the *Termination Competition* since 2004)
- **This talk:** randomized programs $\rightsquigarrow$ **Probabilistic TRSs**

# Deciding $\rho(B_{\mathcal{G}}) \leq 1$

$\rho(B_{\mathcal{G}}) =$ largest absolute value among the eigenvalues (roots of the characteristic polynomial $\chi_{B_{\mathcal{G}}}$).

# Deciding $\rho(B_{\mathcal{G}}) \leq 1$

$\rho(B_{\mathcal{G}}) =$ largest absolute value among the eigenvalues (roots of the characteristic polynomial $\chi_{B_{\mathcal{G}}}$).

## Perron–Frobenius

For $B_{\mathcal{G}} \in \mathbb{R}_{\geq 0}^{n \times n}$, the spectral radius $\rho(B_{\mathcal{G}})$ is itself $\rho(B_{\mathcal{G}}) \in \mathbb{R}_{\geq 0}$ (*real and non-negative*).

# Deciding $\rho(B_{\mathcal{G}}) \leq 1$

$\rho(B_{\mathcal{G}})$ = largest absolute value among the eigenvalues (roots of the characteristic polynomial $\chi_{B_{\mathcal{G}}}$).

## Perron–Frobenius

For $B_{\mathcal{G}} \in \mathbb{R}_{\geq 0}^{n \times n}$, the spectral radius $\rho(B_{\mathcal{G}})$ is itself $\rho(B_{\mathcal{G}}) \in \mathbb{R}_{\geq 0}$ (*real and non-negative*).

Hence $\rho(B_{\mathcal{G}}) > 1$ can be phrased in the **existential theory of the reals** which is decidable:

$$\rho(B_{\mathcal{G}}) > 1 \quad \iff \quad \exists x. \left( x > 1 \ \wedge \ \chi_{B_{\mathcal{G}}}(x) = 0 \right),$$

# Deciding $\rho(B_{\mathcal{G}}) \leq 1$

$\rho(B_{\mathcal{G}})$ = largest absolute value among the eigenvalues (roots of the characteristic polynomial $\chi_{B_{\mathcal{G}}}$).

## Perron–Frobenius

For $B_{\mathcal{G}} \in \mathbb{R}_{\geq 0}^{n \times n}$, the spectral radius $\rho(B_{\mathcal{G}})$ is itself $\rho(B_{\mathcal{G}}) \in \mathbb{R}_{\geq 0}$ (*real and non-negative*).

Hence $\rho(B_{\mathcal{G}}) > 1$ can be phrased in the **existential theory of the reals** which is decidable:

$$\rho(B_{\mathcal{G}}) > 1 \quad \iff \quad \exists x. \left( x > 1 \wedge \chi_{B_{\mathcal{G}}}(x) = 0 \right),$$

## Deciding $\mathtt{AST}$ for an SCFG $\mathcal{G}$

1. $\mathcal{G}$ productive? No $\to$ Reject, Yes $\to$ go on.
2. $\rho(B_{\mathcal{G}}) > 1$? Yes $\to$ Reject, No $\to$ Accept.