Proving Almost-Sure Termination of Probabilistic Programs Using Term Rewriting

Jan-Christoph Kassing, Jürgen Giesl

Research Group Computer Science 2, RWTH Aachen
"Programming Languages and Verification"

September 2023

Termination and Complexity Analysis for Programs

Java

C

Haskell

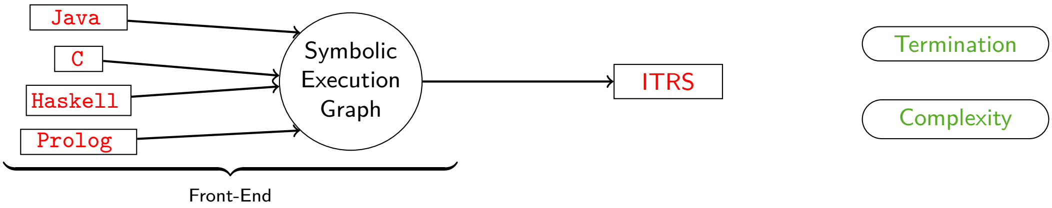
Prolog

Termination

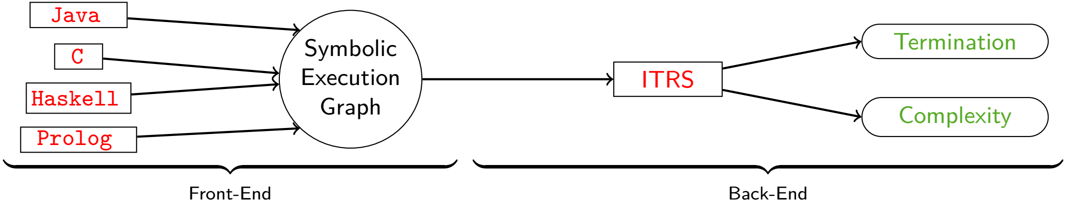
Complexity

Termination and Complexity Analysis for Programs

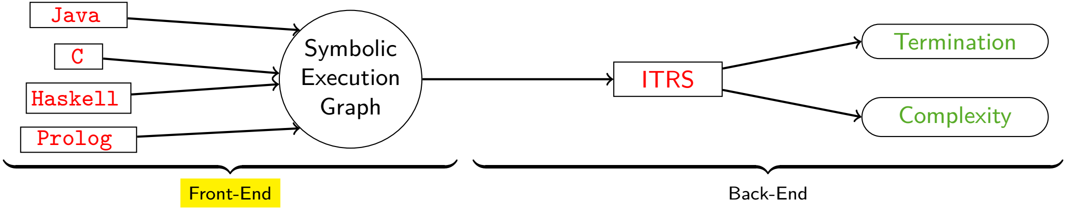
Termination and Complexity Analysis for Programs



Termination and Complexity Analysis for Programs

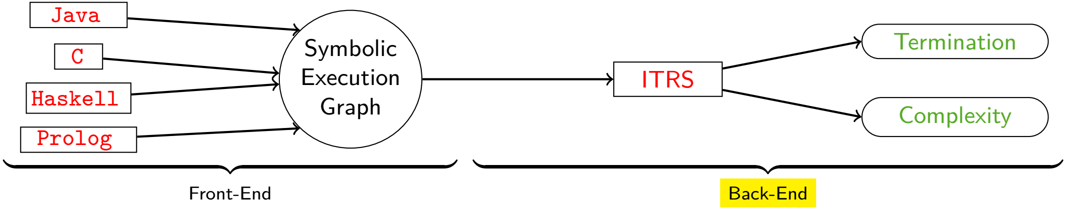


Termination and Complexity Analysis for Programs



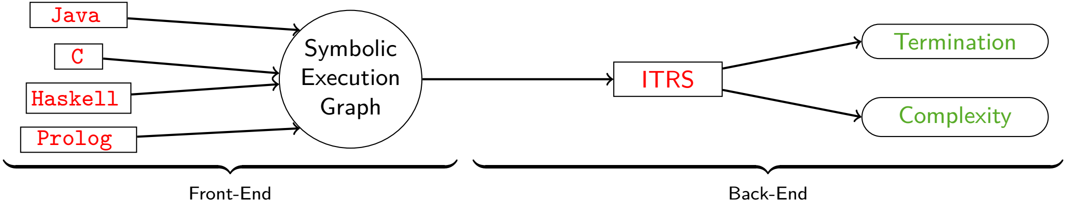
- language-specific features when generating symbolic execution graph

Termination and Complexity Analysis for Programs



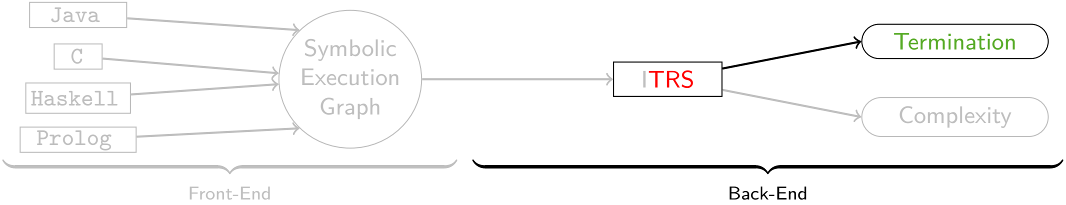
- language-specific features when generating symbolic execution graph
- back-end analyzes Term Rewrite Systems and/or Integer Transition Systems

Termination and Complexity Analysis for Programs

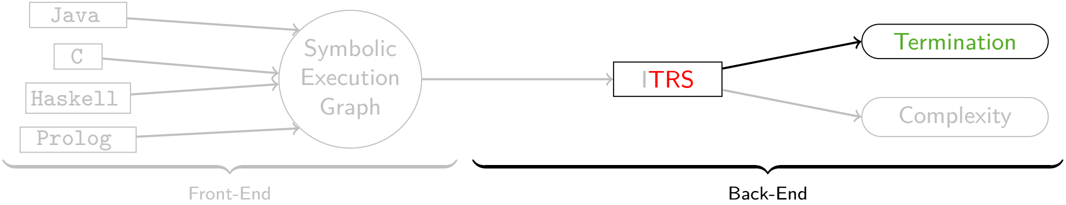


- language-specific features when generating symbolic execution graph
- back-end analyzes Term Rewrite Systems and/or Integer Transition Systems
- powerful termination and complexity analysis implemented in AProVE
  - Termination Competition since 2004 (Java, C, Haskell, Prolog, TRS)
  - SV-COMP since 2014 (C)

Termination of Term Rewrite Systems



Termination of Term Rewrite Systems



- termination analysis for probabilistic TRSs

Automatic Termination Analysis for TRSs

$$\mathcal{R}_{plus}: \quad \begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:
$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \rightarrow & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \rightarrow & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

Computation "2 + 2":

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

$$\mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), \mathsf{s}(\mathsf{s}(\mathcal{O})))$$

Computation "2 + 2":

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \rightarrow & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \rightarrow & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

Computation "2 + 2":

$$\begin{array}{rl} & \mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), \mathsf{s}(\mathsf{s}(\mathcal{O}))) \\ \rightarrow_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{s}(\mathsf{s}(\mathcal{O})))) \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

Computation "2 + 2":

$$\begin{array}{ll} & \mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), \mathsf{s}(\mathsf{s}(\mathcal{O}))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{s}(\mathsf{s}(\mathcal{O})))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{s}(\mathsf{plus}(\mathcal{O}, \mathsf{s}(\mathsf{s}(\mathcal{O}))))) \end{array}$$

Automatic Termination Analysis for TRSs

$$\mathcal{R}_{plus}: \quad \begin{aligned} \mathsf{plus}(\mathcal{O}, y) &\to y \\ \mathsf{plus}(\mathsf{s}(x), y) &\to \mathsf{s}(\mathsf{plus}(x, y)) \end{aligned}$$

Computation "2 + 2":

$$\begin{aligned} & \mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), \mathsf{s}(\mathsf{s}(\mathcal{O}))) \\ \to_{\mathcal{R}_{plus}} \; & \mathsf{s}(\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{s}(\mathsf{s}(\mathcal{O})))) \\ \to_{\mathcal{R}_{plus}} \; & \mathsf{s}(\mathsf{s}(\mathsf{plus}(\mathcal{O}, \mathsf{s}(\mathsf{s}(\mathcal{O}))))) \\ \to_{\mathcal{R}_{plus}} \; & \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathcal{O})))) \end{aligned}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:
$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

Computation "2 + 2":
$$\begin{array}{rl} & \mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), \mathsf{s}(\mathsf{s}(\mathcal{O}))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{s}(\mathsf{s}(\mathcal{O})))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{s}(\mathsf{plus}(\mathcal{O}, \mathsf{s}(\mathsf{s}(\mathcal{O}))))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathcal{O})))) \end{array}$$

Termination: $\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \dots$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

Computation "2 + 2":

$$\begin{array}{rl} & \mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), \mathsf{s}(\mathsf{s}(\mathcal{O}))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{s}(\mathsf{s}(\mathcal{O})))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{s}(\mathsf{plus}(\mathcal{O}, \mathsf{s}(\mathsf{s}(\mathcal{O}))))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathcal{O})))) \end{array}$$

Termination: $\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \ldots$

Goal: Find a well-founded order $\succ$ such that $s \to_{\mathcal{R}} t$ implies $s \succ t$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

Computation "2 + 2":

$$\begin{array}{ll} & \mathsf{plus}(\mathsf{s}(\mathsf{s}(\mathcal{O})), \mathsf{s}(\mathsf{s}(\mathcal{O}))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{s}(\mathsf{s}(\mathcal{O})))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{s}(\mathsf{plus}(\mathcal{O}, \mathsf{s}(\mathsf{s}(\mathcal{O}))))) \\ \to_{\mathcal{R}_{plus}} & \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathcal{O})))) \end{array}$$

Termination: $\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \dots$

Goal: Find a well-founded order $\succ$ such that $s \to_{\mathcal{R}} t$ implies $s \succ t$

Well-Founded: There is no infinite sequence $t_0 \succ t_1 \succ t_2 \succ \dots$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{aligned} \mathsf{plus}(\mathcal{O}, y) &\rightarrow y \\ \mathsf{plus}(\mathsf{s}(x), y) &\rightarrow \mathsf{s}(\mathsf{plus}(x, y)) \end{aligned}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

Goal: Find well-founded order $\succ$ such that $s \to_{\mathcal{R}} t$ implies $s \succ t$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \succ & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \succ & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

Goal: Find well-founded order $\succ$ such that $s \rightarrow_{\mathcal{R}} t$ implies $s \succ t$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \succ & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \succ & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} Pol(\mathsf{plus}(\mathcal{O}, y)) & > & Pol(y) \\ Pol(\mathsf{plus}(\mathsf{s}(x), y)) & > & Pol(\mathsf{s}(\mathsf{plus}(x, y))) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} Pol(\mathsf{plus}(\mathcal{O}, y)) & > & Pol(y) \\ Pol(\mathsf{plus}(\mathsf{s}(x), y)) & > & Pol(\mathsf{s}(\mathsf{plus}(x, y))) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\ldots, x, \ldots) > f_{Pol}(\ldots, y, \ldots)$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} Pol(\mathsf{plus}(\mathcal{O}, y)) & > & Pol(y) \\ Pol(\mathsf{plus}(\mathsf{s}(x), y)) & > & Pol(\mathsf{s}(\mathsf{plus}(x, y))) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\ldots, x, \ldots) > f_{Pol}(\ldots, y, \ldots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x + 1 \\ \mathsf{plus}_{Pol}(x, y) & = & 2x + y + 1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} Pol(\mathsf{plus}(\mathcal{O}, y)) & > & Pol(y) \\ Pol(\mathsf{plus}(\mathsf{s}(x), y)) & > & Pol(\mathsf{s}(\mathsf{plus}(x, y))) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\ldots, x, \ldots) > f_{Pol}(\ldots, y, \ldots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x + 1 \\ \mathsf{plus}_{Pol}(x, y) & = & 2x + y + 1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}_{Pol}(\mathcal{O}_{Pol}, y) & > & y \\ Pol(\mathsf{plus}(\mathsf{s}(x), y)) & > & Pol(\mathsf{s}(\mathsf{plus}(x, y))) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\dots, x, \dots) > f_{Pol}(\dots, y, \dots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x + 1 \\ \mathsf{plus}_{Pol}(x, y) & = & 2x + y + 1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}_{Pol}(0, y) & > & y \\ Pol(\mathsf{plus}(\mathsf{s}(x), y)) & > & Pol(\mathsf{s}(\mathsf{plus}(x, y))) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\ldots, x, \ldots) > f_{Pol}(\ldots, y, \ldots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x + 1 \\ \mathsf{plus}_{Pol}(x, y) & = & 2x + y + 1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} 2 \cdot 0 + y + 1 & > & y \\ Pol(\mathsf{plus}(\mathsf{s}(x), y)) & > & Pol(\mathsf{s}(\mathsf{plus}(x, y))) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\ldots, x, \ldots) > f_{Pol}(\ldots, y, \ldots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x + 1 \\ \mathsf{plus}_{Pol}(x, y) & = & 2x + y + 1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} y+1 & > & y \\ Pol(\mathsf{plus}(\mathsf{s}(x),y)) & > & Pol(\mathsf{s}(\mathsf{plus}(x,y))) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation Pol such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\ldots, x, \ldots) > f_{Pol}(\ldots, y, \ldots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x+1 \\ \mathsf{plus}_{Pol}(x,y) & = & 2x+y+1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} y+1 & > & y \\ Pol(\mathsf{plus}(\mathsf{s}(x), y)) & > & Pol(\mathsf{s}(\mathsf{plus}(x, y))) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\dots, x, \dots) > f_{Pol}(\dots, y, \dots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x+1 \\ \mathsf{plus}_{Pol}(x, y) & = & 2x+y+1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:
$$\begin{array}{rcl} y+1 & > & y \\ \mathsf{plus}_{Pol}(\mathsf{s}_{Pol}(x), y) & > & \mathsf{s}_{Pol}(\mathsf{plus}_{Pol}(x, y)) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\ldots, x, \ldots) > f_{Pol}(\ldots, y, \ldots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x+1 \\ \mathsf{plus}_{Pol}(x, y) & = & 2x+y+1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} y+1 & > & y \\ \mathsf{plus}_{Pol}(x+1,y) & > & \mathsf{s}_{Pol}(2x+y+1) \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\ldots, x, \ldots) > f_{Pol}(\ldots, y, \ldots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x+1 \\ \mathsf{plus}_{Pol}(x,y) & = & 2x+y+1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} y+1 & > & y \\ 2(x+1)+y+1 & > & (2x+y+1)+1 \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\ldots, x, \ldots) > f_{Pol}(\ldots, y, \ldots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x+1 \\ \mathsf{plus}_{Pol}(x,y) & = & 2x+y+1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} y+1 & > & y \\ 2x+y+3 & > & 2x+y+2 \end{array}$$

Goal: Find monotonic, natural polynomial interpretation Pol such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\ldots, x, \ldots) > f_{Pol}(\ldots, y, \ldots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x+1 \\ \mathsf{plus}_{Pol}(x,y) & = & 2x+y+1 \end{array}$$

Automatic Termination Analysis for TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} y+1 & > & y \\ 2x+y+3 & > & 2x+y+2 \end{array}$$

Goal: Find monotonic, natural polynomial interpretation *Pol* such that

$$\ell \to r \in \mathcal{R} \text{ implies } Pol(\ell) > Pol(r)$$

- natural: $Pol(f) = f_{Pol}$ a polynomial with coefficients $\in \mathbb{N}$
- monotonic: if $x > y$, then $f_{Pol}(\dots, x, \dots) > f_{Pol}(\dots, y, \dots)$

$$\begin{array}{rcl} \mathcal{O}_{Pol} & = & 0 \\ \mathsf{s}_{Pol}(x) & = & x+1 \\ \mathsf{plus}_{Pol}(x,y) & = & 2x+y+1 \end{array}$$

$\Rightarrow$ proves termination

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \qquad \mathsf{g}(x) \;\to\; \{\, \tfrac{1}{2} : x,\; \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \qquad \mathsf{g}(x) \to \{ \tfrac{1}{2} : x,\ \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \}$$

Distribution: $\{ p_1 : t_1, \ldots, p_k : t_k \}$ with $p_1 + \ldots + p_k = 1$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \qquad \mathsf{g}(x) \;\to\; \{\, \tfrac{1}{2} : x,\; \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$\{\, 1 : \mathsf{g}(\mathcal{O}) \,\}$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \quad \mathsf{g}(x) \to \{\, \tfrac{1}{2} : x,\ \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\{\, 1 : \mathsf{g}(\mathcal{O}) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, {}^1\!/_2 : \mathcal{O},\ {}^1\!/_2 : \mathsf{g}^2(\mathcal{O}) \,\}$$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \quad \mathsf{g}(x) \to \{ \tfrac{1}{2} : x, \ \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \}$$

Distribution: $\{ p_1 : t_1, \ldots, p_k : t_k \}$ with $p_1 + \ldots + p_k = 1$

$$\{ 1 : \mathsf{g}(\mathcal{O}) \}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : \mathcal{O}, \ 1/2 : \mathsf{g}^2(\mathcal{O}) \}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : \mathcal{O}, \ 1/4 : \mathsf{g}(\mathcal{O}), \ 1/4 : \mathsf{g}^3(\mathcal{O}) \}$$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \quad \mathsf{g}(x) \to \{\, \tfrac{1}{2} : x,\; \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\begin{aligned}
& \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \; & \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_2 : \mathsf{g}^2(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \; & \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_4 : \mathsf{g}(\mathcal{O}),\; {}^1\!/_4 : \mathsf{g}^3(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \; & \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_8 : \mathcal{O},\; {}^1\!/_8 : \mathsf{g}^2(\mathcal{O}),
\end{aligned}$$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \quad \mathsf{g}(x) \to \{\, \tfrac{1}{2} : x,\ \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\begin{aligned}
& \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, 1/2 : \mathcal{O},\ 1/2 : \mathsf{g}^2(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, 1/2 : \mathcal{O},\ 1/4 : \mathsf{g}(\mathcal{O}),\ 1/4 : \mathsf{g}^3(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, 1/2 : \mathcal{O},\ 1/8 : \mathcal{O},\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^4(\mathcal{O}) \,\}
\end{aligned}$$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \qquad \mathsf{g}(x) \;\to\; \{\, \tfrac{1}{2} : x,\; \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x))\,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$
\begin{aligned}
& \{\, 1 : \mathsf{g}(\mathcal{O})\,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_2 : \mathsf{g}^2(\mathcal{O})\,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_4 : \mathsf{g}(\mathcal{O}),\; {}^1\!/_4 : \mathsf{g}^3(\mathcal{O})\,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_8 : \mathcal{O},\; {}^1\!/_8 : \mathsf{g}^2(\mathcal{O}),\; {}^1\!/_8 : \mathsf{g}^2(\mathcal{O}),\; {}^1\!/_8 : \mathsf{g}^4(\mathcal{O})\,\}
\end{aligned}
$$

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \quad \mathsf{g}(x) \to \{\, \tfrac{1}{2} : x,\ \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\begin{aligned}
& \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, 1/2 : \mathcal{O},\ 1/2 : \mathsf{g}^2(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, 1/2 : \mathcal{O},\ 1/4 : \mathsf{g}(\mathcal{O}),\ 1/4 : \mathsf{g}^3(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, 1/2 : \mathcal{O},\ 1/8 : \mathcal{O},\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^4(\mathcal{O}) \,\}
\end{aligned}$$

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$ No

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \qquad \mathsf{g}(x) \;\to\; \{\, \tfrac{1}{2} : x,\; \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$
\begin{aligned}
& \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \;& \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_2 : \mathsf{g}^2(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \;& \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_4 : \mathsf{g}(\mathcal{O}),\; {}^1\!/_4 : \mathsf{g}^3(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \;& \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_8 : \mathcal{O},\; {}^1\!/_8 : \mathsf{g}^2(\mathcal{O}),\; {}^1\!/_8 : \mathsf{g}^2(\mathcal{O}),\; {}^1\!/_8 : \mathsf{g}^4(\mathcal{O}) \,\}
\end{aligned}
$$

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$ No
- $\mathcal{R}$ is *almost-surely terminating (AST)* iff $\lim_{n\to\infty} |\mu_n| = 1$ for every infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \qquad \mathsf{g}(x) \;\to\; \{\, \tfrac{1}{2} : x,\; \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$|\mu|$

$$
\begin{aligned}
& \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^{1}/_{2} : \mathcal{O},\; {}^{1}/_{2} : \mathsf{g}^2(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^{1}/_{2} : \mathcal{O},\; {}^{1}/_{4} : \mathsf{g}(\mathcal{O}),\; {}^{1}/_{4} : \mathsf{g}^3(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^{1}/_{2} : \mathcal{O},\; {}^{1}/_{8} : \mathcal{O},\; {}^{1}/_{8} : \mathsf{g}^2(\mathcal{O}),\; {}^{1}/_{8} : \mathsf{g}^2(\mathcal{O}),\; {}^{1}/_{8} : \mathsf{g}^4(\mathcal{O}) \,\}
\end{aligned}
$$

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$ No
- $\mathcal{R}$ is *almost-surely terminating (AST)* iff $\lim_{n\to\infty} |\mu_n| = 1$ for every infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \qquad \mathsf{g}(x) \;\to\; \{\, \tfrac{1}{2} : x,\; \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$|\mu|$

$$\{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \qquad 0$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_2 : \mathsf{g}^2(\mathcal{O}) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_4 : \mathsf{g}(\mathcal{O}),\; {}^1\!/_4 : \mathsf{g}^3(\mathcal{O}) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, {}^1\!/_2 : \mathcal{O},\; {}^1\!/_8 : \mathcal{O},\; {}^1\!/_8 : \mathsf{g}^2(\mathcal{O}),\; {}^1\!/_8 : \mathsf{g}^2(\mathcal{O}),\; {}^1\!/_8 : \mathsf{g}^4(\mathcal{O}) \,\}$$

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$ No
- $\mathcal{R}$ is *almost-surely terminating (AST)* iff $\lim_{n\to\infty} |\mu_n| = 1$ for every infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \quad g(x) \to \{ \tfrac{1}{2} : x, \; \tfrac{1}{2} : g(g(x)) \}$$

Distribution: $\{ p_1 : t_1, \ldots, p_k : t_k \}$ with $p_1 + \ldots + p_k = 1$

		$\lvert\mu\rvert$
	$\{ 1 : g(\mathcal{O}) \}$	0
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{ 1/2 : \mathcal{O}, \; 1/2 : g^2(\mathcal{O}) \}$	1/2
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{ 1/2 : \mathcal{O}, \; 1/4 : g(\mathcal{O}), \; 1/4 : g^3(\mathcal{O}) \}$	
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{ 1/2 : \mathcal{O}, \; 1/8 : \mathcal{O}, \; 1/8 : g^2(\mathcal{O}), \; 1/8 : g^2(\mathcal{O}), \; 1/8 : g^4(\mathcal{O}) \}$	

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$ — No
- $\mathcal{R}$ is *almost-surely terminating (AST)* iff $\lim_{n\to\infty} \lvert\mu_n\rvert = 1$ for every infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \quad \mathsf{g}(x) \to \{\tfrac{1}{2}: x,\ \tfrac{1}{2}: \mathsf{g}(\mathsf{g}(x))\}$$

Distribution: $\{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$

		$\lvert\mu\rvert$
	$\{1 : \mathsf{g}(\mathcal{O})\}$	0
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{1/2 : \mathcal{O},\ 1/2 : \mathsf{g}^2(\mathcal{O})\}$	1/2
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{1/2 : \mathcal{O},\ 1/4 : \mathsf{g}(\mathcal{O}),\ 1/4 : \mathsf{g}^3(\mathcal{O})\}$	1/2
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{1/2 : \mathcal{O},\ 1/8 : \mathcal{O},\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^4(\mathcal{O})\}$	

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$ No
- $\mathcal{R}$ is *almost-surely terminating (AST)*
  iff $\lim_{n\to\infty} \lvert\mu_n\rvert = 1$ for every infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \quad \mathsf{g}(x) \to \{\tfrac{1}{2} : x,\ \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x))\}$$

Distribution: $\{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \ldots + p_k = 1$

		$\lvert\mu\rvert$
	$\{1 : \mathsf{g}(\mathcal{O})\}$	0
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{1/2 : \mathcal{O},\ 1/2 : \mathsf{g}^2(\mathcal{O})\}$	$1/2$
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{1/2 : \mathcal{O},\ 1/4 : \mathsf{g}(\mathcal{O}),\ 1/4 : \mathsf{g}^3(\mathcal{O})\}$	$1/2$
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{1/2 : \mathcal{O},\ 1/8 : \mathcal{O},\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^4(\mathcal{O})\}$	$5/8$

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$ — No
- $\mathcal{R}$ is *almost-surely terminating (AST)* iff $\lim_{n\to\infty} \lvert\mu_n\rvert = 1$ for every infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \quad g(x) \to \{\, \tfrac{1}{2} : x,\ \tfrac{1}{2} : g(g(x)) \,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

		$\lvert\mu\rvert$
	$\{\, 1 : g(\mathcal{O}) \,\}$	0
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{\, 1/2 : \mathcal{O},\ 1/2 : g^2(\mathcal{O}) \,\}$	$1/2$
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{\, 1/2 : \mathcal{O},\ 1/4 : g(\mathcal{O}),\ 1/4 : g^3(\mathcal{O}) \,\}$	$1/2$
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{\, 1/2 : \mathcal{O},\ 1/8 : \mathcal{O},\ 1/8 : g^2(\mathcal{O}),\ 1/8 : g^2(\mathcal{O}),\ 1/8 : g^4(\mathcal{O}) \,\}$	$5/8$

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$ — No
- $\mathcal{R}$ is *almost-surely terminating (AST)* iff $\lim_{n\to\infty} \lvert\mu_n\rvert = 1$ for every infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$ — Yes

Termination of Probabilistic TRSs [CADE23]

$$\mathcal{R}_{rw}: \qquad \mathsf{g}(x) \;\to\; \{\, \tfrac{1}{2} : x,\ \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Let *Pol* be a multilinear natural monotonic polynomial interpretation.
For all $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{R}$ let

- $Pol(\ell) > Pol(r_j)$ for some $1 \leq j \leq k$
- $Pol(\ell) \geq p_1 \cdot Pol(r_1) + \ldots + p_k \cdot Pol(r_k)$

Then $\mathcal{R}$ is AST.

Termination of Probabilistic TRSs [CADE23]

$$\mathcal{R}_{rw}: \qquad g(x) \;\to\; \{\, \tfrac{1}{2} : x,\; \tfrac{1}{2} : g(g(x)) \,\}$$

Let *Pol* be a multilinear natural monotonic polynomial interpretation.
For all $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{R}$ let

- $Pol(\ell) > Pol(r_j)$ for some $1 \leq j \leq k$
- $Pol(\ell) \geq p_1 \cdot Pol(r_1) + \ldots + p_k \cdot Pol(r_k)$

Then $\mathcal{R}$ is AST.

- multilinear: monomials like $x \cdot y$, but no monomials like x^2

Termination of Probabilistic TRSs [CADE23]

$$\mathcal{R}_{rw}: \qquad g(x) \;\to\; \{\, \tfrac{1}{2} : x,\; \tfrac{1}{2} : g(g(x)) \,\}$$

Let *Pol* be a multilinear natural monotonic polynomial interpretation.
For all $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{R}$ let

- $Pol(\ell) > Pol(r_j)$ for some $1 \leq j \leq k$
- $Pol(\ell) \geq p_1 \cdot Pol(r_1) + \ldots + p_k \cdot Pol(r_k)$

Then $\mathcal{R}$ is AST.

- multilinear: monomials like $x \cdot y$, but no monomials like x^2

$$g_{Pol}(x) \;=\; 1 + x$$

Termination of Probabilistic TRSs [CADE23]

$$\mathcal{R}_{rw}: \quad \mathsf{g}(x) \to \{\tfrac{1}{2} : x,\ \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x))\}$$

Let *Pol* be a multilinear natural monotonic polynomial interpretation.
For all $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{R}$ let

- $Pol(\ell) > Pol(r_j)$ for some $1 \leq j \leq k$
- $Pol(\ell) \geq p_1 \cdot Pol(r_1) + \ldots + p_k \cdot Pol(r_k)$

Then $\mathcal{R}$ is AST.

- multilinear: monomials like $x \cdot y$, but no monomials like x^2

$$\mathsf{g}_{Pol}(x) = 1 + x$$

Termination of Probabilistic TRSs [CADE23]

$$\mathcal{R}_{rw}: \quad 1 + x \quad \to \quad \{ \tfrac{1}{2} : x, \ \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \}$$

Let *Pol* be a multilinear natural monotonic polynomial interpretation.
For all $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{R}$ let

- $Pol(\ell) > Pol(r_j)$ for some $1 \leq j \leq k$
- $Pol(\ell) \geq p_1 \cdot Pol(r_1) + \ldots + p_k \cdot Pol(r_k)$

Then $\mathcal{R}$ is AST.

- multilinear: monomials like $x \cdot y$, but no monomials like x^2

$$\mathsf{g}_{Pol}(x) \quad = \quad 1 + x$$

Termination of Probabilistic TRSs [CADE23]

$$\mathcal{R}_{rw}: \quad 1 + x \quad \geq \quad \{\, \tfrac{1}{2} : x,\ \tfrac{1}{2} : \mathsf{g}(\mathsf{g}(x)) \,\}$$

Let *Pol* be a multilinear natural monotonic polynomial interpretation.
For all $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{R}$ let

- $Pol(\ell) > Pol(r_j)$ for some $1 \leq j \leq k$
- $Pol(\ell) \geq p_1 \cdot Pol(r_1) + \ldots + p_k \cdot Pol(r_k)$

Then $\mathcal{R}$ is AST.

- multilinear: monomials like $x \cdot y$, but no monomials like x^2

$$\mathsf{g}_{Pol}(x) \quad = \quad 1 + x$$

Termination of Probabilistic TRSs [CADE23]

$$\mathcal{R}_{rw}: \quad 1 + x \quad \geq \quad \frac{1}{2} \cdot x + \frac{1}{2} \cdot (2 + x)$$

Let *Pol* be a multilinear natural monotonic polynomial interpretation.
For all $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{R}$ let

- $Pol(\ell) > Pol(r_j)$ for some $1 \leq j \leq k$
- $Pol(\ell) \geq p_1 \cdot Pol(r_1) + \ldots + p_k \cdot Pol(r_k)$

Then $\mathcal{R}$ is AST.

- multilinear: monomials like $x \cdot y$, but no monomials like x^2

$$g_{Pol}(x) \quad = \quad 1 + x$$

Termination of Probabilistic TRSs [CADE23]

$$\mathcal{R}_{rw}: \quad 1 + x \quad \geq \; 1 + x$$

Let *Pol* be a multilinear natural monotonic polynomial interpretation.
For all $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{R}$ let

- $Pol(\ell) > Pol(r_j)$ for some $1 \leq j \leq k$
- $Pol(\ell) \geq p_1 \cdot Pol(r_1) + \ldots + p_k \cdot Pol(r_k)$

Then $\mathcal{R}$ is AST.

- multilinear: monomials like $x \cdot y$, but no monomials like x^2

$$g_{Pol}(x) \quad = \quad 1 + x$$

Termination of Probabilistic TRSs [CADE23]

$$\mathcal{R}_{rw}: \quad 1 + x \quad \geq \; 1 + x$$

Let *Pol* be a multilinear natural monotonic polynomial interpretation.
For all $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{R}$ let

- $Pol(\ell) > Pol(r_j)$ for some $1 \leq j \leq k$
- $Pol(\ell) \geq p_1 \cdot Pol(r_1) + \ldots + p_k \cdot Pol(r_k)$

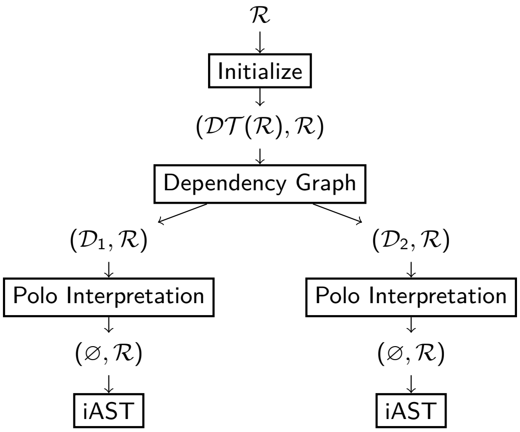
Then $\mathcal{R}$ is AST.

- multilinear: monomials like $x \cdot y$, but no monomials like x^2

$$g_{Pol}(x) \quad = \quad 1 + x$$

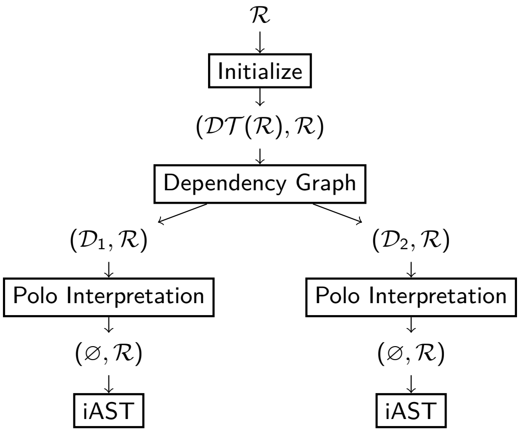
$\Rightarrow$ proves AST

Probabilistic DT Framework [CADE23]



DT framework for *innermost* AST of PTRSs

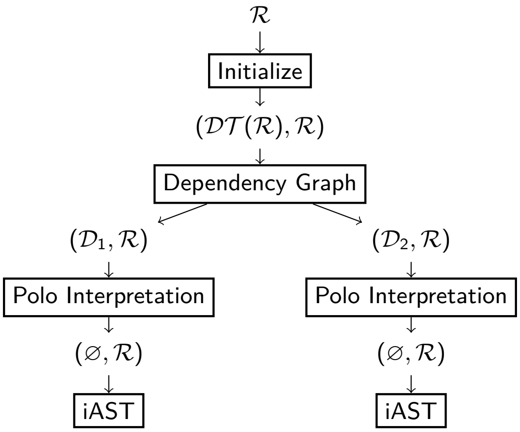
Probabilistic DT Framework [CADE23]



DT framework for *innermost* AST of PTRSs

- allows for modular termination proofs

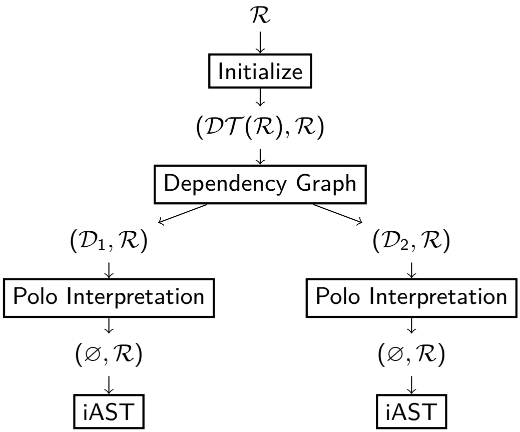
Probabilistic DT Framework [CADE23]



DT framework for *innermost* AST of PTRSs

- allows for modular termination proofs
- focus on innermost evaluation

Probabilistic DT Framework [CADE23]



DT framework for *innermost* AST of PTRSs

- allows for modular termination proofs
- focus on innermost evaluation
- developed multiple different processors
  - Dependency Graph Processor
  - Reduction Pair Processor
  - Usable Rules Processor
  - Usable Terms Processor
  - Probability Removal Processor
  - ...