Dependency Pairs for Expected Innermost Runtime Complexity and Strong Almost-Sure Termination of Probabilistic Term Rewriting

Jan-Christoph Kassing, Leon Spitzer, and Jürgen Giesl
RWTH Aachen University
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Complexity of TRSs

$\mathcal{R}_{\mathsf{double}}$:

$$\begin{aligned} \mathsf{double}(0) &\to 0 \\ \mathsf{double}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) \end{aligned}$$

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$$\mathsf{double}(0) \to 0$$
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$\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))$

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$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0))))$$

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$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0))$$

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Termination

TRS $\mathcal{R}$ is *terminating* if there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$

$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0)) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(0)$$

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Runtime Complexity, $\mathrm{rc}_{\mathcal{R}}$

$$\mathrm{rc}_{\mathcal{R}}(n) = \sup\{\mathrm{dh}_{\mathcal{R}}(t) \mid \qquad |t| \leq n\}$$

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derivation **h**eight

$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0)) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(0)$$

$\mathrm{dh}_{\mathcal{R}}\,(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) =$ "max number of steps" $= 3$

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Innermost Runtime Complexity, $\mathrm{rc}_{\mathcal{R}}$

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Basic Terms, $\mathcal{T}_B$

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Defined Symbols Σ_D: double,

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Defined Symbols Σ_D: double, Constructors Σ_C: $\mathsf{s}, 0$

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$$\mathrm{rc}_{\mathcal{R}}(n) = \sup\{\mathrm{dh}_{\mathcal{R}}(t) \mid t \in \mathcal{T}_B, |t| \leq n\}$$

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Basic Terms, $\mathcal{T}_B$

Terms $f(c_1, \ldots, c_n) \in \mathcal{T}_B$ are *basic* if $f \in \Sigma_D$ and all c_i are constructor terms.

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$\mathrm{dh}_{\mathcal{R}}(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) = \text{“max number of steps”} = 3$ $\mathrm{rc}_{\mathcal{R}_{\mathsf{double}}}(n) = n - 1$

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$\mathrm{dh}_{\mathcal{R}}\,(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) = \text{“max number of steps”} = 3$ $\quad \mathrm{rc}_{\mathcal{R}_{\mathsf{double}}}(n) = n - 1 \in \mathcal{O}(n^1)$

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$\mathrm{dh}_{\mathcal{R}}\,(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) = \text{“max number of steps”} = 3$ $\mathrm{rc}_{\mathcal{R}_{\mathsf{double}}}(n) = n - 1 \in \mathcal{O}(n^1)$ $\mathrm{rc}_{\mathcal{R}_{\mathsf{double}}} = \mathrm{Pol}_1$

Basic Terms, $\mathcal{T}_B$

Terms $f(c_1, \ldots, c_n) \in \mathcal{T}_B$ are *basic* if $f \in \Sigma_D$ and all c_i are constructor terms.

Termination and Complexity Analysis for Programs

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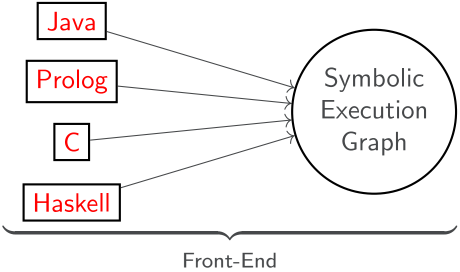
Java

Prolog

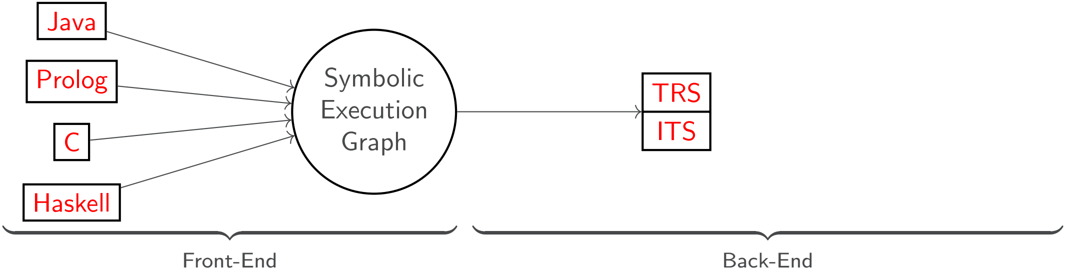
C

Haskell

Termination and Complexity Analysis for Programs



Termination and Complexity Analysis for Programs



Java

Prolog

C

Haskell

Symbolic Execution Graph

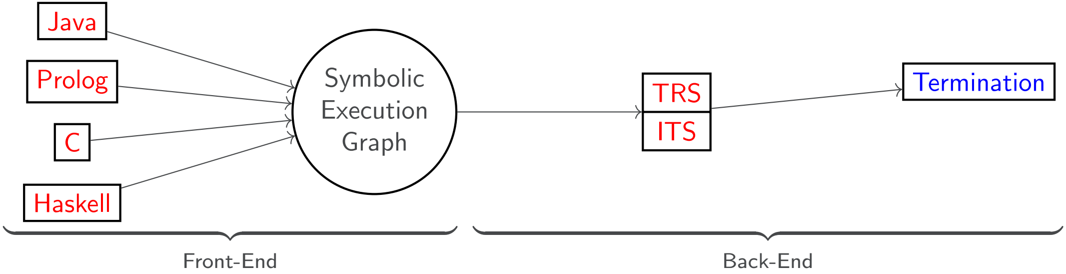
TRS

ITS

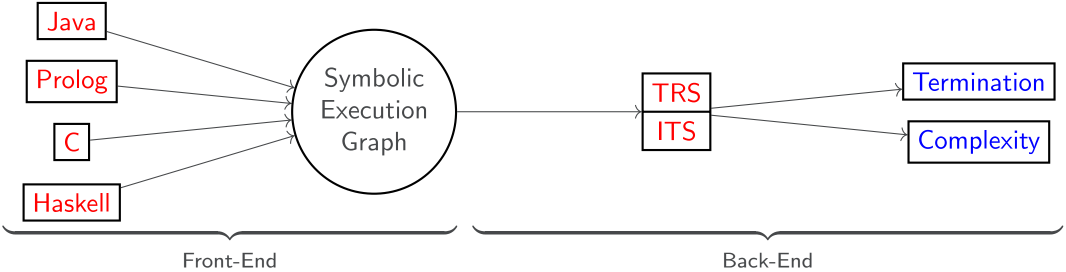
Front-End

Back-End

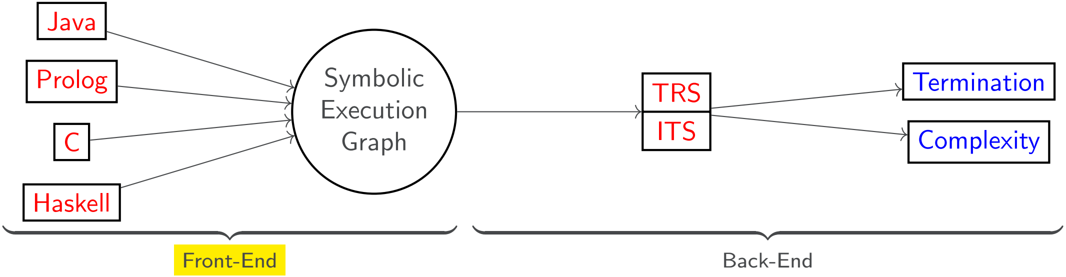
Termination and Complexity Analysis for Programs



Termination and Complexity Analysis for Programs

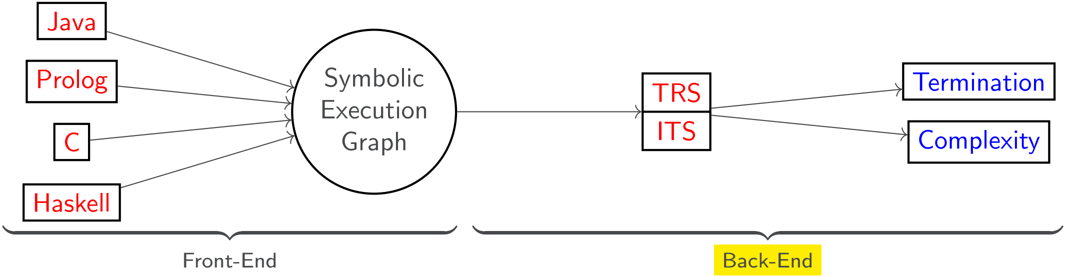


Termination and Complexity Analysis for Programs



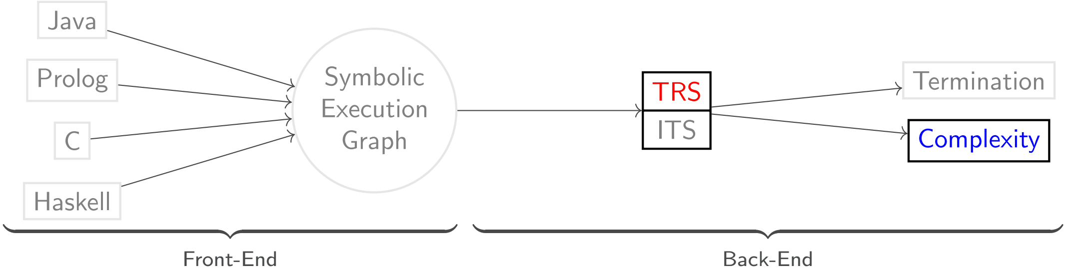
- language-specific features when generating symbolic execution graph

Termination and Complexity Analysis for Programs



- language-specific features when generating symbolic execution graph
- back-end analyzes Term Rewrite Systems and/or Integer Transition Systems

Termination and Complexity Analysis for Programs



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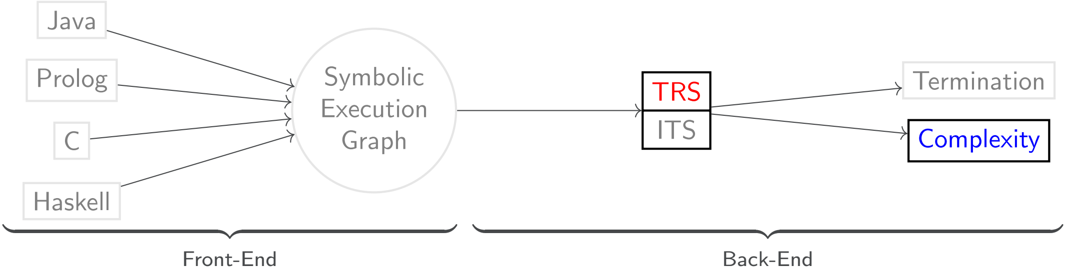
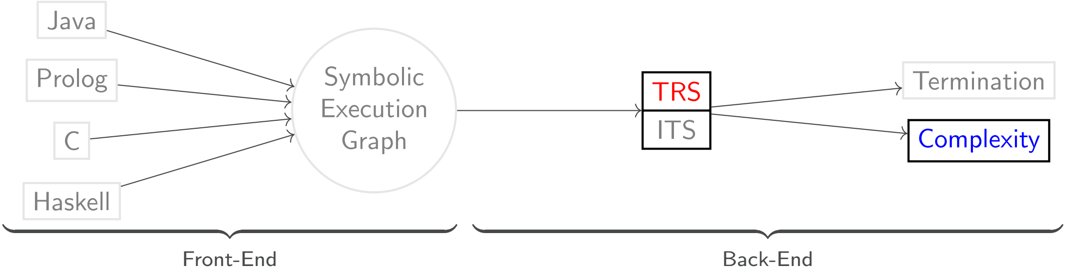


- Proving Termination and Complexity of TRSs

Termination and Complexity Analysis for Programs



- Proving Termination and Complexity of TRSs
- Proving Termination and Complexity of Probabilistic TRSs

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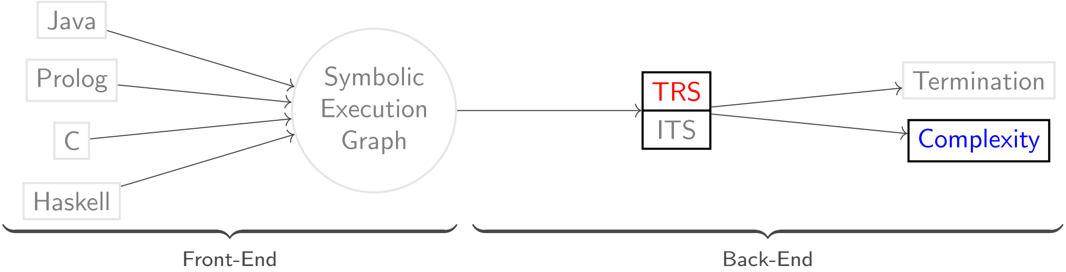

- Proving Termination and Complexity of TRSs
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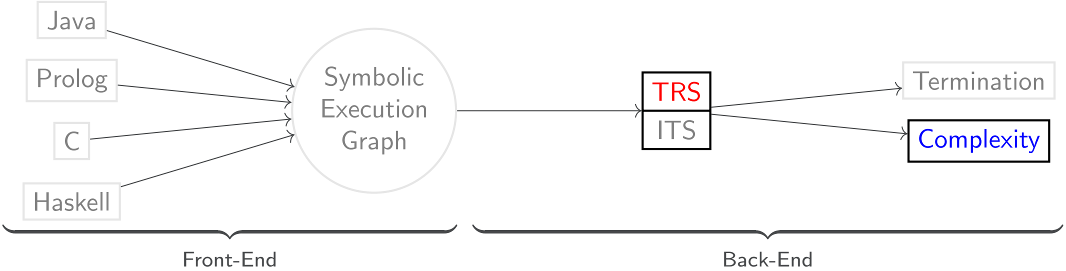
- Dependency Pairs for Complexity Analysis of Probabilistic TRSs

Termination and Complexity Analysis for Programs



- Proving Termination and Complexity of TRSs
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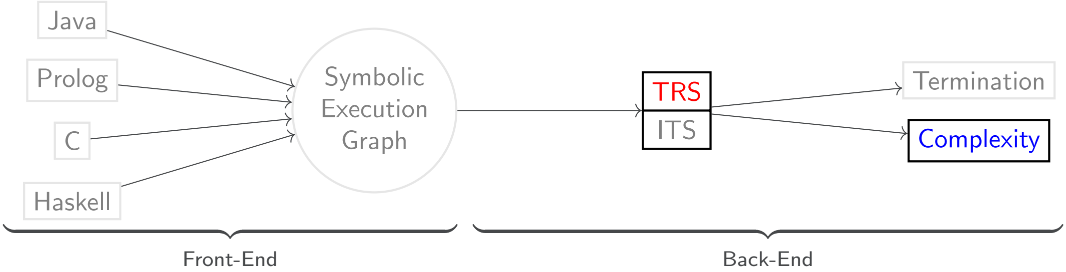
Termination and Complexity Analysis for Programs



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Proving Termination

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Natural & Monotonic Polynomial Interpretation $\mathcal{I}$

- natural: $\mathcal{I}_f(x_1, \ldots, x_n)$ is a polynomial with natural coefficients for every function symbol $f \in \Sigma$
- monotonic: $x > y$ implies $\mathcal{I}_f(\ldots, x, \ldots) > \mathcal{I}_f(\ldots, y, \ldots)$ for every function symbol $f \in \Sigma$

Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
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$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

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Proving Termination With Natural & Monotonic $\mathcal{I}$ [Lankford'79]

$\mathcal{R}$ is terminating if for all rules $\ell \to r : \mathcal{I}(\ell) > \mathcal{I}(r)$

Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

$$\mathcal{I}(\mathsf{double}(0)) > \mathcal{I}(0)$$
$$\mathcal{I}(\mathsf{double}(\mathsf{s}(x))) > \mathcal{I}(\mathsf{s}(\mathsf{s}(\mathsf{double}(x))))$$

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Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

$$2 \cdot \mathcal{I}(0) + 1 > 1$$
$$2 \cdot \mathcal{I}(\mathsf{s}(x)) + 1 > \mathcal{I}(\mathsf{s}(\mathsf{double}(x))) + 1$$

$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

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$\mathcal{R}$ is terminating if for all rules $\ell \to r : \mathcal{I}(\ell) > \mathcal{I}(r)$

Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

$$2 \cdot 1 + 1 > 1$$

$$2 \cdot (x+1) + 1 > 2 \cdot x + 2$$

$$\mathcal{I}_{0} = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

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Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

$$3 > 1$$

$$2x + 3 > 2x + 2$$

$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

Proving Termination With Natural & Monotonic $\mathcal{I}$ [Lankford'79]

$\mathcal{R}$ is terminating if for all rules $\ell \to r : \mathcal{I}(\ell) > \mathcal{I}(r)$

Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

$$3 > 1$$
$$2x + 3 > 2x + 2$$

$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

Proving Termination With Natural & Monotonic $\mathcal{I}$ [Lankford'79]

$\mathcal{R}$ is terminating if for all rules $\ell \to r : \mathcal{I}(\ell) > \mathcal{I}(r)$

$$t_0 \quad \overset{\mathsf{i}}{\to}_{\mathcal{R}_{\mathsf{double}}} \quad t_1 \quad \overset{\mathsf{i}}{\to}_{\mathcal{R}_{\mathsf{double}}} \quad t_2 \quad \overset{\mathsf{i}}{\to}_{\mathcal{R}_{\mathsf{double}}} \quad \ldots$$

Proving Termination

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$$2x + 3 > 2x + 2$$

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$$\mathcal{I}(t_0) \quad > \quad \mathcal{I}(t_1) \quad > \quad \mathcal{I}(t_2) \quad > \quad \ldots$$

Proving Complexity

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(\mathsf{0}) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{double}(x))$$

$$\mathcal{I}_{\mathsf{0}} = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = \; x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

Proving Complexity

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Goal: Infer complexity from the highest degree of $\mathcal{I}$ [Hofbauer&Lautemann'89]

Proving Complexity

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For basic term $t = \mathsf{double}(\mathsf{s}^n(0))$:

Proving Complexity

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For basic term $t = \mathsf{double}(\mathsf{s}^n(0))$: $\qquad \mathcal{I}(t) = 2 \cdot (n + 1) + 1$

Proving Complexity

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⇝ at most linear runtime complexity

Proving Complexity

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$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{double}(x))$$

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Proving Complexity

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Complexity Polynomial Interpretation (CPI)

$\mathcal{I}_f(x_1, \ldots, x_n) = a_1 x_1 + \ldots + a_n x_n + b$ for every constructor $f \in \Sigma_C$ with $b \in \mathbb{N}, a_i \in \{0, 1\}$

Proving Complexity

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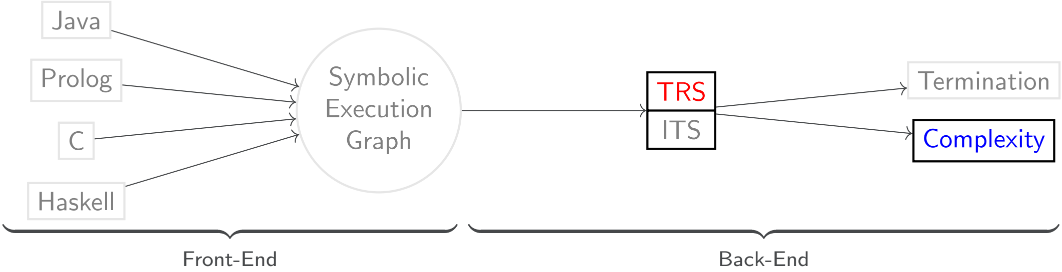
Termination and Complexity Analysis for Programs



- Proving Termination and Complexity of TRSs
- Proving Termination and Complexity of Probabilistic TRSs
- Dependency Pairs for Complexity Analysis of Probabilistic TRSs

Expected Runtime of Probabilistic TRSs

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \left\{ \frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head} \right\}$

Expected Runtime of Probabilistic TRSs

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \{\frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head}\}$

Multidistribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Runtime of Probabilistic TRSs

$\mathcal{R}_{\text{coin}}$: $\text{coin} \to \left\{ \frac{9}{10} : \text{coin}, \frac{1}{10} : \text{head} \right\}$

Multidistribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

$\mathfrak{T}$: $\boxed{1 \mid \text{coin}}$

Expected Runtime of Probabilistic TRSs

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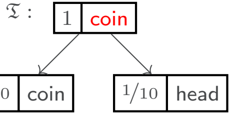

Expected Runtime of Probabilistic TRSs

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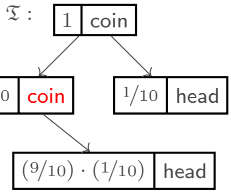

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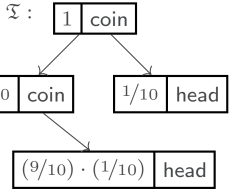


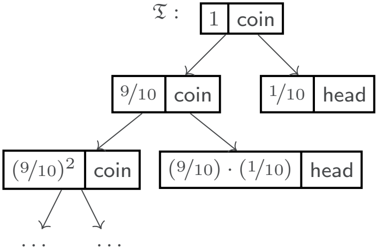
Expected Runtime of Probabilistic TRSs

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \left\{ \frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head} \right\}$

Multidistribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T})$$



Expected Runtime of Probabilistic TRSs

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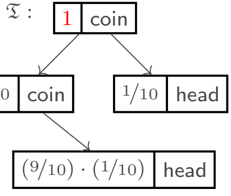
Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 +$$



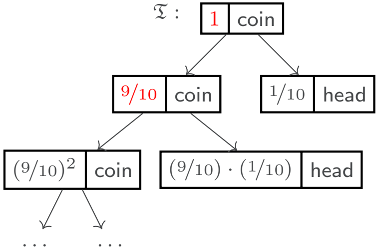
Expected Runtime of Probabilistic TRSs

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \left\{ \frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head} \right\}$

Multidistribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} +$$



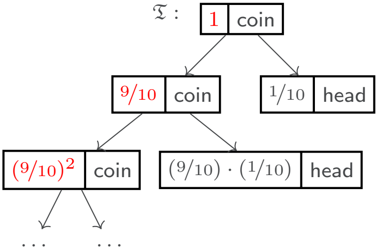
Expected Runtime of Probabilistic TRSs

$\mathcal{R}_{\mathsf{coin}}$: coin $\to \left\{ \frac{9}{10} : \text{coin}, \frac{1}{10} : \text{head} \right\}$

Multidistribution: $\mu = \{p_1 : t_1, \dots, p_k : t_k\}$ with $p_1 + \dots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + (\frac{9}{10})^2 + \dots$$



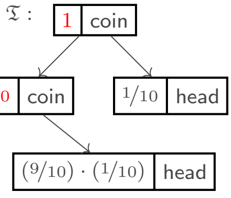
Expected Runtime of Probabilistic TRSs

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \{ \frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head} \}$

Multidistribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + (\frac{9}{10})^2 + \ldots = \sum_{n=0}^{\infty} (\frac{9}{10})^n = 10$$



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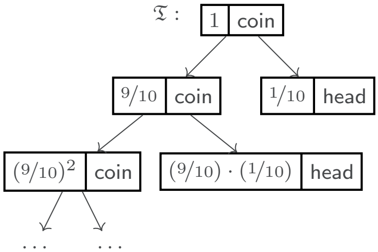
Expected Derivation Height:



Expected Runtime of Probabilistic TRSs

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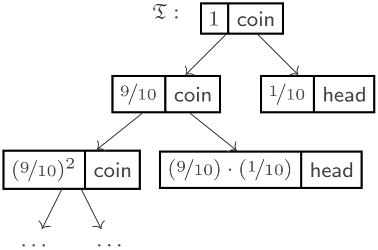
Expected Derivation Height:

$$\mathrm{edh}_{\mathcal{R}_{\mathsf{coin}}}(\mathsf{coin}) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with } \mathsf{coin}\}$$



Expected Runtime of Probabilistic TRSs

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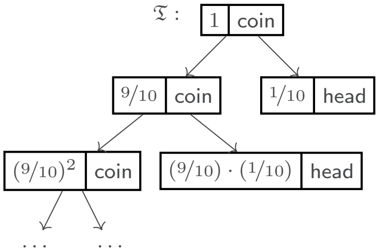

Expected Runtime of Probabilistic TRSs

$\mathcal{R}_{\text{coin}}$: $\text{coin} \to \{\frac{9}{10} : \text{coin}, \frac{1}{10} : \text{head}\}$

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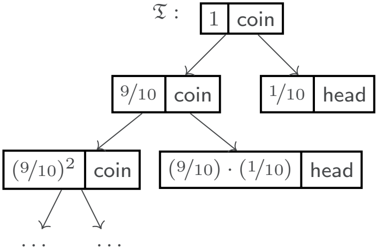
$$\text{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + (\frac{9}{10})^2 + \dots = \sum_{n=0}^{\infty} (\frac{9}{10})^n = 10$$

Expected Derivation Height:

$$\text{edh}_{\mathcal{R}_{\text{coin}}}(\text{coin}) = \sup\{\text{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with coin}\} = 10$$

Expected Runtime Complexity:



Expected Runtime of Probabilistic TRSs

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \{ \frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head} \}$

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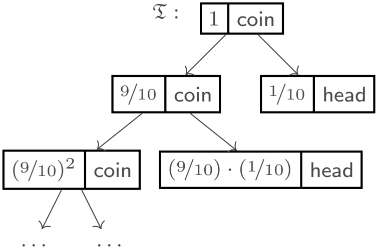
$$\mathrm{edh}_{\mathcal{R}_{\mathsf{coin}}}(\mathsf{coin}) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with } \mathsf{coin}\} = 10$$

Expected Runtime Complexity:

$$\mathrm{erc}_{\mathcal{R}_{\mathsf{coin}}}(n) = \sup\{\mathrm{edh}_{\mathcal{R}_{\mathsf{coin}}}(t) \mid t \in \mathcal{T}_B, |t| \le n\}$$



Expected Runtime of Probabilistic TRSs

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \{\frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head}\}$ constant complexity: Pol_0

Multidistribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

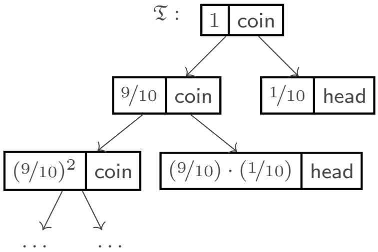
$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + (\frac{9}{10})^2 + \ldots = \sum_{n=0}^{\infty} (\frac{9}{10})^n = 10$$

Expected Derivation Height:

$$\mathrm{edh}_{\mathcal{R}_{\mathsf{coin}}}(\mathsf{coin}) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with } \mathsf{coin}\} = 10$$

Expected Runtime Complexity:

$$\mathrm{erc}_{\mathcal{R}_{\mathsf{coin}}}(n) = \sup\{\mathrm{edh}_{\mathcal{R}_{\mathsf{coin}}}(t) \mid t \in \mathcal{T}_B, |t| \le n\}$$



Expected Runtime of Probabilistic TRSs

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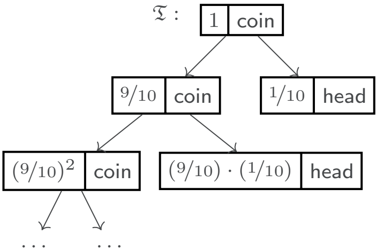

Expected Derivation Length:

$$\text{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + (\frac{9}{10})^2 + \dots = \sum_{n=0}^{\infty} (\frac{9}{10})^n = 10$$

Expected Derivation Height:

$$\text{edh}_{\mathcal{R}_{\text{coin}}}(\text{coin}) = \sup\{\text{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with coin}\} = 10$$

Expected Runtime Complexity:

$$\text{erc}_{\mathcal{R}_{\text{coin}}}(n) = \sup\{\text{edh}_{\mathcal{R}_{\text{coin}}}(t) \mid t \in \mathcal{T}_B, |t| \leq n\}$$

Strong Almost-Sure Termination (SAST) [Avanzini&Dal Lago&Yamada'20]

PTRS $\mathcal{R}$ is SAST if $\text{edh}_{\mathcal{R}}(t)$ is finite for every start term t.

Proving Expected Runtime Complexity & SAST

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \left\{ \frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head} \right\}$

Proving Expected Runtime Complexity & SAST

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \left\{ \frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head} \right\}$

Multilinear Polynomials

$x \cdot y$ is multilinear but x^2 is not.

Proving Expected Runtime Complexity & SAST

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \left\{\frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head}\right\}$

$$\mathcal{I}_{\mathsf{coin}} = 1 \qquad \mathcal{I}_{\mathsf{head}} = 0$$

Multilinear Polynomials

$x \cdot y$ is multilinear but x^2 is not.

Proving Expected Runtime Complexity & `SAST`

$\mathcal{R}_{\mathsf{coin}}$: $\mathsf{coin} \to \{\frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head}\}$

$\mathcal{I}_{\mathsf{coin}} = 1 \qquad \mathcal{I}_{\mathsf{head}} = 0$

Theorem: Natural & Monotonic & Multilinear $\mathcal{I}$ [Avanzini&Dal Lago&Yamada'20]

$\mathcal{R}$ is `SAST` if for all rules $\ell \to \mu$: $\mathcal{I}(\ell) > \mathbb{E}_{\mathcal{I}}(\mu)$

Proving Expected Runtime Complexity & SAST

$\mathcal{R}_{\mathsf{coin}}$: $\quad \mathsf{coin} \to \left\{ \frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head} \right\}$

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Theorem: Natural & Monotonic & Multilinear $\mathcal{I}$ [Avanzini&Dal Lago&Yamada'20]

$\mathcal{R}$ is SAST if for all rules $\ell \to \mu$: $\mathcal{I}(\ell) > \mathbb{E}_{\mathcal{I}}(\mu) = \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(r_j)$ for $\mu = \{p_1 : r_1, \ldots, p_k : r_k\}$

Proving Expected Runtime Complexity & SAST

$\mathcal{R}_{\mathsf{coin}}$: $\mathcal{I}(\mathsf{coin}) > \mathbb{E}_{\mathcal{I}}(\{\frac{9}{10} : \mathsf{coin}, \frac{1}{10} : \mathsf{head}\})$

$\mathcal{I}_{\mathsf{coin}} = 1 \qquad \mathcal{I}_{\mathsf{head}} = 0$

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Proving Expected Runtime Complexity & SAST

$\mathcal{R}_{\mathsf{coin}}$: $\mathcal{I}(\mathsf{coin}) > \frac{9}{10} \cdot \mathcal{I}(\mathsf{coin}) + \frac{1}{10} \cdot \mathcal{I}(\mathsf{head})$

$\mathcal{I}_{\mathsf{coin}} = 1$ $\mathcal{I}_{\mathsf{head}} = 0$

Theorem: Natural & Monotonic & Multilinear $\mathcal{I}$ [Avanzini&Dal Lago&Yamada'20]

$\mathcal{R}$ is `SAST` if for all rules $\ell \to \mu$: $\mathcal{I}(\ell) > \mathbb{E}_{\mathcal{I}}(\mu) = \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(r_j)$ for $\mu = \{p_1 : r_1, \ldots, p_k : r_k\}$

Proving Expected Runtime Complexity & SAST

$\mathcal{R}_{\mathsf{coin}}$: $1 > \frac{9}{10}$

$$\mathcal{I}_{\mathsf{coin}} = 1 \qquad \mathcal{I}_{\mathsf{head}} = 0$$

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Proving Expected Runtime Complexity & SAST

$\mathcal{R}_{\mathsf{coin}}$: $1 > \frac{9}{10}$

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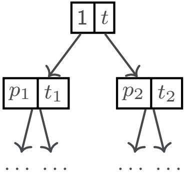
Theorem: Natural & Monotonic & Multilinear $\mathcal{I}$ [Avanzini&Dal Lago&Yamada'20]

$\mathcal{R}$ is SAST if for all rules $\ell \to \mu$: $\mathcal{I}(\ell) > \mathbb{E}_{\mathcal{I}}(\mu) = \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(r_j)$ for $\mu = \{p_1 : r_1, \ldots, p_k : r_k\}$



Proving Expected Runtime Complexity & $\mathtt{SAST}$

$\mathcal{R}_{\mathsf{coin}}$: $$1 > \frac{9}{10}$$

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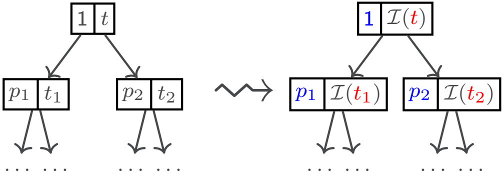

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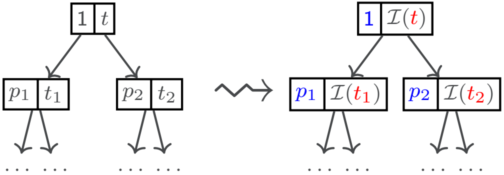

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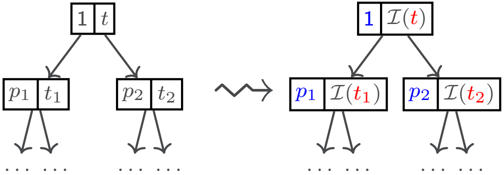

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Proving Expected Runtime Complexity & `SAST`

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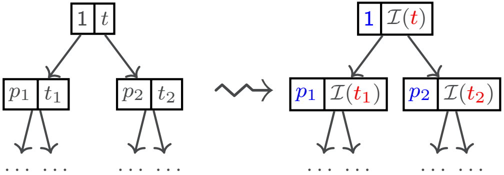

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Proving Expected Runtime Complexity & SAST

$\mathcal{R}_{\mathsf{coin}}$: $\quad 1 > \frac{9}{10} \quad \epsilon = {}^{1}/{}_{10}$

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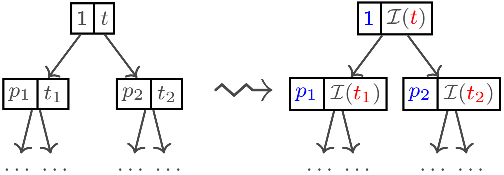

$$\mathbb{E}_{\mathcal{I}}(\mu_0) = 1 \cdot \mathcal{I}(t)$$

$$> \epsilon \cdot 1$$

ϵ - minimal decrease for all rules

$$\mathbb{E}_{\mathcal{I}}(\mu_1) = p_1 \cdot \mathcal{I}(t_1) + p_2 \cdot \mathcal{I}(t_2)$$

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Proving Expected Runtime Complexity & SAST

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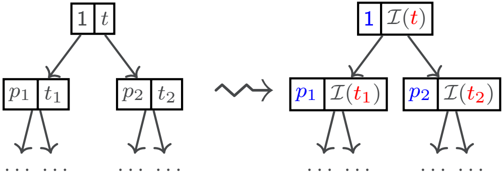

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$> \epsilon \cdot 1$ ϵ - minimal decrease for all rules

$\mathbb{E}_{\mathcal{I}}(\mu_1) = p_1 \cdot \mathcal{I}(t_1) + p_2 \cdot \mathcal{I}(t_2)$

$> \epsilon \cdot (p_1 + p_2)$

$\ldots$

Proving Expected Runtime Complexity & SAST

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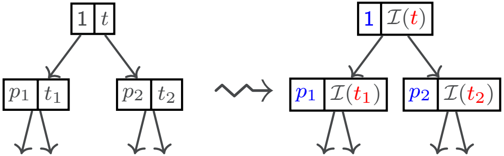

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Goal: Infer expected complexity from the highest degree of $\mathcal{I}$.

Proving Expected Runtime Complexity & $\mathtt{SAST}$

$\mathcal{R}_{\text{coin}}$: $1 > \frac{9}{10}$ $\epsilon = {}^{1}\!/_{10}$

$\mathcal{I}_{\text{coin}} = 1$ $\mathcal{I}_{\text{head}} = 0$ $\rightsquigarrow$ constant complexity: Pol_0

Theorem: Natural & Monotonic & Multilinear $\mathcal{I}$ [Avanzini&Dal Lago&Yamada'20]

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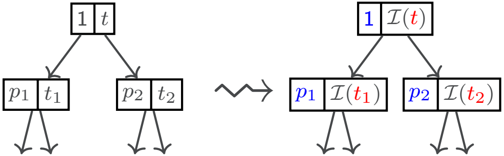

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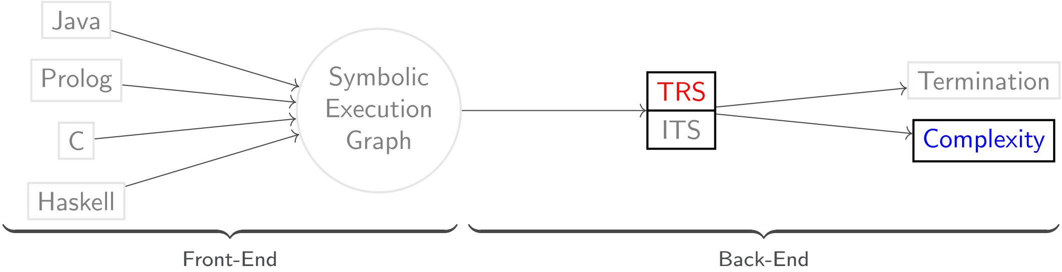
Goal: Infer expected complexity from the highest degree of $\mathcal{I}$.

- Restrict to basic start terms and CPI

Termination and Complexity Analysis for Programs



- Proving Termination and Complexity of TRSs
- Proving Termination and Complexity of Probabilistic TRSs
- Dependency Pairs for Complexity Analysis of Probabilistic TRSs

Annotated Dependency Pairs (ADPs)

$\mathcal{R}_{\mathsf{geo}}$:

$$\begin{aligned}
\mathsf{start}(x,y) &\to \{1 : \mathsf{q}(\mathsf{geo}(x),y,y)\}\\
\mathsf{geo}(x) &\to \{{}^{1}/{}_{2} : \mathsf{geo}(\mathsf{s}(x)),\ {}^{1}/{}_{2} : x\}\\
\mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) &\to \{1 : \mathsf{q}(x,y,z)\}\\
\mathsf{q}(x,0,\mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}(x,\mathsf{s}(z),\mathsf{s}(z)))\}\\
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\end{aligned}$$

Annotated Dependency Pairs (ADPs)

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\end{aligned}
$$

$$\boxed{1 \,\big|\, \mathsf{geo}(x)}$$

Annotated Dependency Pairs (ADPs)

$\mathcal{R}_{\mathsf{geo}}$:

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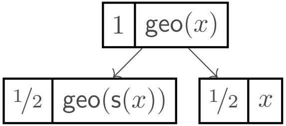

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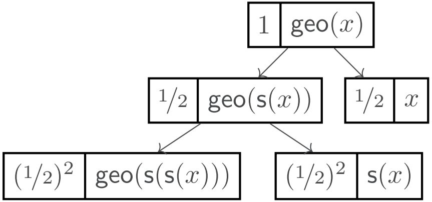

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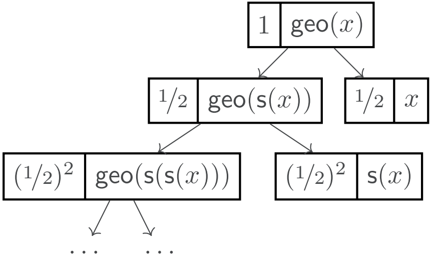

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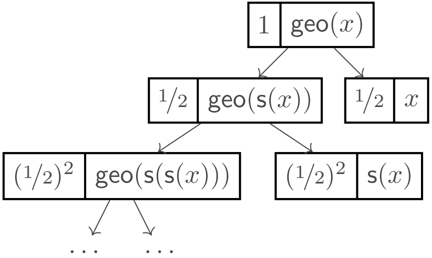

$\mathsf{geo}(x) \sim$ geometric distribution

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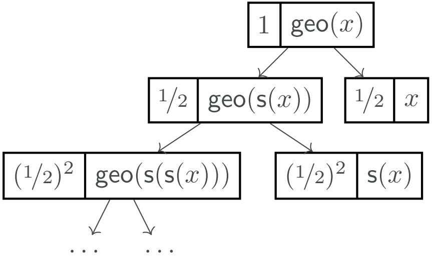

1 | $\mathsf{geo}(x)$

$1/2$ | $\mathsf{geo}(\mathsf{s}(x))$

$1/2$ | x

$(1/2)^2$ | $\mathsf{geo}(\mathsf{s}(\mathsf{s}(x)))$

$(1/2)^2$ | $\mathsf{s}(x)$

... ...

$\mathsf{geo}(x) \sim$ geometric distribution

⇝ expected constant complexity: Pol_0

Annotated Dependency Pairs (ADPs)

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$\mathsf{q}(x,y,y) \sim \lfloor \frac{x}{y} \rfloor$ (integer division)

Annotated Dependency Pairs (ADPs)

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$\mathsf{q}(x,y,y) \sim \lfloor \frac{x}{y} \rfloor$ (integer division)

$\rightsquigarrow$ expected linear complexity: Pol_1

Annotated Dependency Pairs (ADPs)

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Defined Symbols Σ_D: $\mathsf{start}, \mathsf{geo}, \mathsf{q}$

Annotated Dependency Pairs (ADPs)

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\end{aligned}$$

Defined Symbols Σ_D: $\mathsf{start}, \mathsf{geo}, \mathsf{q}$ Constructors Σ_C: $0, \mathsf{s}$

Annotated Dependency Pairs (ADPs)

$\mathcal{R}_{\mathsf{geo}}$:

$$\begin{aligned}
\mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}\\
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Defined Symbols Σ_D: $\mathsf{start}, \mathsf{geo}, \mathsf{q}$ Constructors Σ_C: $0, \mathsf{s}$

1. “$\sharp \sim$ functions calls that count for complexity”

Annotated Dependency Pairs (ADPs)

$\mathcal{ADP}(\mathcal{R}_{\mathsf{geo}})$:

$$\begin{aligned}
\mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{true}}\\
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\end{aligned}$$

Defined Symbols Σ_D: $\mathsf{start}, \mathsf{geo}, \mathsf{q}$ Constructors Σ_C: $0, \mathsf{s}$

1. "$\sharp \sim$ functions calls that count for complexity"
2. "$\{\ldots\}^{\mathsf{true}} \sim$ rule can be used below $\sharp$"

Annotated Dependency Pairs (ADPs)

$\mathcal{ADP}(\mathcal{R}_{\mathsf{geo}})$:

$$\begin{aligned}
\mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{true}}\\
\mathsf{geo}(x) &\to \{{}^1\!/_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/_2 : x\}^{\mathsf{true}}\\
\mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) &\to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{true}}\\
\mathsf{q}(x,0,\mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{true}}\\
\mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) &\to \{1 : 0\}^{\mathsf{true}}
\end{aligned}$$

Defined Symbols Σ_D: $\mathsf{start},\mathsf{geo},\mathsf{q}$ Constructors Σ_C: $0,\mathsf{s}$

1. "$\sharp \sim$ functions calls that count for complexity"
2. "$\{\ldots\}^{\mathsf{true}} \sim$ rule can be used below $\sharp$"
3. ADP Problem $\langle\mathcal{P},\mathcal{S},\mathcal{K}\rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P}$ set of all ADPs
   - $\mathcal{S}$ set of ADPs which we count for complexity
   - $\mathcal{K}$ set of ADPs whose complexity we already considered

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle
\end{tikzcd}$$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
    - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs

$$\begin{tikzcd}
\text{PTRS } \mathcal{R} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle
\end{tikzcd}$$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
    - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
    - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity

$$\begin{tikzcd} \text{PTRS } \mathcal{R} \arrow[d, "\text{Chain Criterion}"'] \\ \langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle \end{tikzcd}$$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered

$$\text{PTRS } \mathcal{R} \xrightarrow{\text{Chain Criterion}} \langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle$$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem

$$\begin{tikzcd}
\text{PTRS } \mathcal{R} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle
\end{tikzcd}$$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem

$$\begin{tikzcd}
\text{PTRS } \mathcal{R} \arrow[d, swap, "\text{Chain Criterion}"] \\
\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle \arrow[d, swap, "\mathrm{Proc}_A"] \arrow[d, "\mathrm{Pol}_{m_1}"] \\
\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle
\end{tikzcd}$$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem

$$\begin{tikzcd}
\text{PTRS } \mathcal{R} \arrow[d, swap, "\text{Chain Criterion}"] \\
\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle \arrow[d, swap, "\mathrm{Proc}_A"] \arrow[d, "\mathrm{Pol}_{m_1}"] \\
\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle \arrow[d, swap, "\mathrm{Proc}_B"] \arrow[d, "\mathrm{Pol}_{m_2}"] \\
\ldots
\end{tikzcd}$$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
    - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
    - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
    - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem

Sound:
Complexity of $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$

$$\begin{array}{ccc} & \text{PTRS } \mathcal{R} & \\ \text{Chain Criterion} & \downarrow & \\ & \langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle & \\ \mathrm{Proc}_A & \downarrow & \mathrm{Pol}_{m_1} \\ & \langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle & \\ \mathrm{Proc}_B & \downarrow & \mathrm{Pol}_{m_2} \\ & \ldots & \end{array}$$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem

Sound:
Complexity of $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$
$\leq$ (is bounded by)
$\max\{$Complexity of $\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle, \mathrm{Pol}_{m_1}\}$
$+$ all previously derived Pol_{m_i}

$$\begin{tikzcd}
\text{PTRS } \mathcal{R} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_A"', "\mathrm{Pol}_{m_1}"] \\
\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle \arrow[d, "\mathrm{Proc}_B"', "\mathrm{Pol}_{m_2}"] \\
\cdots
\end{tikzcd}$$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem
3. Result with solved ADP problem $\mathcal{P} = \mathcal{K}$

$$\begin{tikzcd} \text{PTRS } \mathcal{R} \arrow[d, "\text{Chain Criterion}"'] \\ \langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_A"', "\mathrm{Pol}_{m_1}"] \\ \langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle \arrow[d, "\mathrm{Proc}_B"', "\mathrm{Pol}_{m_2}"] \\ \cdots \end{tikzcd}$$

Sound:

Complexity of $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$

$\leq$ (is bounded by)

$\max\{$Complexity of $\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle, \mathrm{Pol}_{m_1}\}$

$+$ all previously derived Pol_{m_i}

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem
3. Result with solved ADP problem $\mathcal{P} = \mathcal{K}$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_A"', "\mathrm{Pol}_{m_1}"] \\
\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle \arrow[d, "\mathrm{Proc}_B"', "\mathrm{Pol}_{m_2}"] \\
\cdots \arrow[d, "\mathrm{Proc}_C"', "\mathrm{Pol}_{m_3}"] \\
\langle \mathcal{P}'', \varnothing, \mathcal{P}'' \rangle
\end{tikzcd}$$

Sound:

Complexity of $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$

$\leq$ (is bounded by)

$\max\{$Complexity of $\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle, \mathrm{Pol}_{m_1}\}$

$+$ all previously derived Pol_{m_i}

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem
3. Result with solved ADP problem $\mathcal{P} = \mathcal{K}$

Sound:

Complexity of $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$

$\leq$ (is bounded by)

$\max\{$Complexity of $\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle, \mathrm{Pol}_{m_1}\}$

$+$ all previously derived Pol_{m_i}

$$\begin{tikzcd}
\text{PTRS } \mathcal{R} \arrow[d, "\text{Chain Criterion}"'] & \\
\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_A"', "\mathrm{Pol}_{m_1}"] & \\
\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle \arrow[d, "\mathrm{Proc}_B"', "\mathrm{Pol}_{m_2}"] & \\
\cdots \arrow[d, "\mathrm{Proc}_C"', "\mathrm{Pol}_{m_3}"] & \\
\boxed{\langle \mathcal{P}'', \varnothing, \mathcal{P}'' \rangle} & {} \arrow[uuuu, "\max"']
\end{tikzcd}$$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem
3. Result with solved ADP problem $\mathcal{P} = \mathcal{K}$

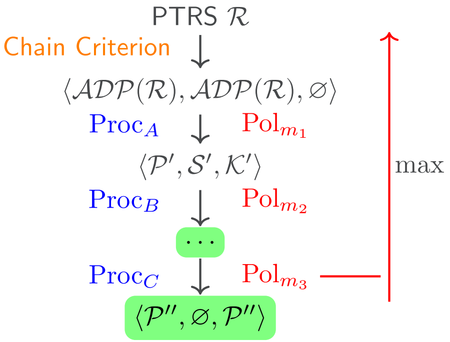
Sound:

Complexity of $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$

$\leq$ (is bounded by)

$\max\{$Complexity of $\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle, \mathrm{Pol}_{m_1}\}$

$+$ all previously derived Pol_{m_i}



PTRS $\mathcal{R}$

Chain Criterion $\downarrow$

$\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle$

$\mathrm{Proc}_A \downarrow \mathrm{Pol}_{m_1}$

$\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle$

$\mathrm{Proc}_B \downarrow \mathrm{Pol}_{m_2}$

$\cdots$

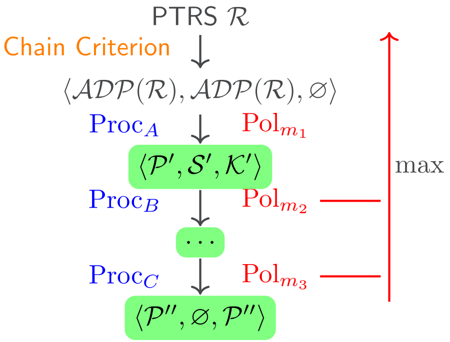
$\mathrm{Proc}_C \downarrow \mathrm{Pol}_{m_3}$

$\langle \mathcal{P}'', \varnothing, \mathcal{P}'' \rangle$

$\max$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
    - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
    - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
    - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem
3. Result with solved ADP problem $\mathcal{P} = \mathcal{K}$



Sound:

Complexity of $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$

$\leq$ (is bounded by)

$\max\{$Complexity of $\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle, \mathrm{Pol}_{m_1}\}$

$+$ all previously derived Pol_{m_i}

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
    - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
    - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
    - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem
3. Result with solved ADP problem $\mathcal{P} = \mathcal{K}$

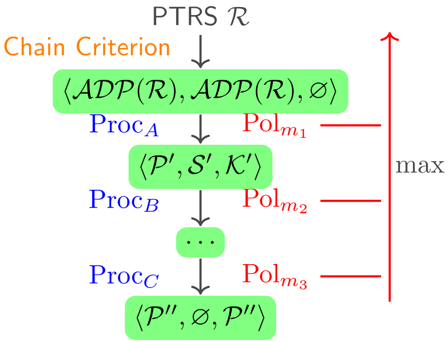
Sound:

Complexity of $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$

$\leq$ (is bounded by)

$\max\{$Complexity of $\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle, \mathrm{Pol}_{m_1}\}$

$+$ all previously derived Pol_{m_i}



PTRS $\mathcal{R}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle$

Proc_A Pol_{m_1}

$\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle$

Proc_B Pol_{m_2}

$\cdots$

Proc_C Pol_{m_3}

$\langle \mathcal{P}'', \varnothing, \mathcal{P}'' \rangle$

$\max$

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem
3. Result with solved ADP problem $\mathcal{P} = \mathcal{K}$

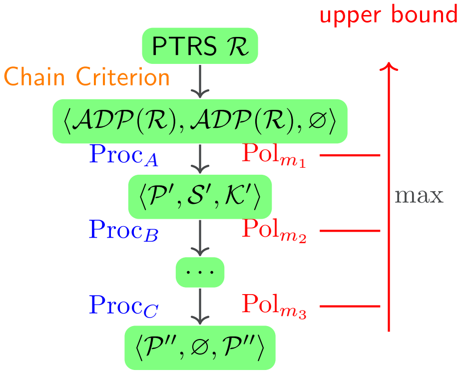
Sound:

Complexity of $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$

$\leq$ (is bounded by)

$\max\{$Complexity of $\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle, \mathrm{Pol}_{m_1}\}$

$+$ all previously derived Pol_{m_i}



upper bound

PTRS $\mathcal{R}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle$

Proc_A Pol_{m_1}

$\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle$

Proc_B Pol_{m_2}

$\cdots$

Proc_C Pol_{m_3}

$\langle \mathcal{P}'', \varnothing, \mathcal{P}'' \rangle$

max

Annotated Dependency Pair Framework

1. Transform PTRS $\mathcal{R}$ into ADP Problem $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$, $\mathcal{P} = \mathcal{S} \uplus \mathcal{K}$
   - $\mathcal{P} = \mathcal{ADP}(\mathcal{R})$ set of all ADPs
   - $\mathcal{S} = \mathcal{ADP}(\mathcal{R})$ set of ADPs which we count for complexity
   - $\mathcal{K} = \varnothing$ set of ADPs whose complexity we already considered
2. Apply Processors to simplify the ADP problem
3. Result with solved ADP problem $\mathcal{P} = \mathcal{K}$

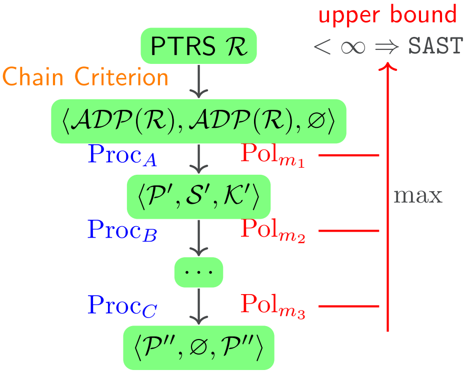
Sound:

Complexity of $\langle \mathcal{P}, \mathcal{S}, \mathcal{K} \rangle$

$\leq$ (is bounded by)

$\max\{$Complexity of $\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle, \mathrm{Pol}_{m_1}\}$

$+$ all previously derived Pol_{m_i}



upper bound

$< \infty \Rightarrow \mathtt{SAST}$

PTRS $\mathcal{R}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle$

Proc_A Pol_{m_1}

$\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle$

Proc_B Pol_{m_2}

$\cdots$

Proc_C Pol_{m_3}

$\langle \mathcal{P}'', \varnothing, \mathcal{P}'' \rangle$

max

Usable Rules Processor

$$
\begin{aligned}
(1)\ \mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{true}} \\
(2)\ \mathsf{geo}(x) &\to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}
\end{aligned}
$$

$$
\begin{aligned}
(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) &\to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{true}} \\
(4)\ \mathsf{q}(x, 0, \mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{true}} \\
(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) &\to \{1 : 0\}^{\mathsf{true}}
\end{aligned}
$$

PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion

$$\downarrow$$

$$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$$

Usable Rules Processor

$$\begin{array}{lrl}(1) & \mathsf{start}(x,y) & \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{true}} \\ (2) & \mathsf{geo}(x) & \to \{\nicefrac{1}{2} : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ \nicefrac{1}{2} : x\}^{\mathsf{true}}\end{array}$$

$$\begin{array}{lrl}(3) & \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) & \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{true}} \\ (4) & \mathsf{q}(x,0,\mathsf{s}(z)) & \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{true}} \\ (5) & \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) & \to \{1 : 0\}^{\mathsf{true}}\end{array}$$

$$\begin{tikzcd}\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\ \langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle\end{tikzcd}$$

Usable Rules

Rules are usable if they can evaluate below an annotated symbol $f^{\#}$.

Usable Rules Processor

$$
\begin{array}{ll}
(1) & \mathsf{start}(x, y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{true}} \\
(2) & \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}
\end{array}
$$

$$
\begin{array}{ll}
(3) & \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{true}} \\
(4) & \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{true}} \\
(5) & \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{true}}
\end{array}
$$

$$
\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle
\end{tikzcd}
$$

Usable Rules

Rules are usable if they can evaluate below an annotated symbol $f^{\#}$.

Usable Rules Processor

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{true}}$$
$$(2)\ \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{true}}$$
$$(4)\ \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{true}}$$
$$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{true}}$$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle
\end{tikzcd}$$

Usable Rules

Rules are usable if they can evaluate below an annotated symbol $f^\#$.

Usable Rules Processor

$$\begin{array}{ll}(1) & \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{true}} \\ (2) & \mathsf{geo}(x) \to \{{}^1\!/\!{}_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/\!{}_2 : x\}^{\mathsf{true}}\end{array}$$

$$\begin{array}{ll}(3) & \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{true}} \\ (4) & \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{true}} \\ (5) & \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{true}}\end{array}$$

$$\begin{tikzcd} \text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\ \langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle \end{tikzcd}$$

$\mathrm{Proc}_{\mathtt{UR}}$ - “Remove flags of unusable rules”

Usable Rules

Rules are usable if they can evaluate below an annotated symbol $f^{\#}$.

Usable Rules Processor

$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{true}}$

$(2)\ \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$

$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{true}}$

$(4)\ \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{true}}$

$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{true}}$

PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion ↓

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ - "Remove flags of unusable rules"

Usable Rules

Rules are usable if they can evaluate below an annotated symbol $f^{\#}$.

Usable Rules Processor

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}$$
$$(2)\ \mathsf{geo}(x) \to \{{}^1\!/_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/_2 : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}$$
$$(4)\ \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}$$
$$(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle
\end{tikzcd}$$

$\mathrm{Proc}_{\mathtt{UR}}$ - “Remove flags of unusable rules”

Usable Rules

Rules are usable if they can evaluate below an annotated symbol $f^{\#}$.

Usable Rules Processor

(1) $\mathsf{start}(x, y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}$
(2) $\mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$

(3) $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$
(4) $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$
(5) $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{UR}}"', "\mathrm{Pol}_0"] \\
\langle \{(1), (2), (3), (4), (5)\}, \{(1), (2), (3), (4), (5)\}, \varnothing \rangle
\end{tikzcd}$$

$\mathrm{Proc}_{\mathtt{UR}}$ - “Remove flags of unusable rules”

Usable Rules

Rules are usable if they can evaluate below an annotated symbol $f^{\#}$.

Dependency Graph Processor

$$\begin{aligned}
(1)\ \mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}\\
(2)\ \mathsf{geo}(x) &\to \{{}^1\!/\!{}_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/\!{}_2 : x\}^{\mathsf{true}}
\end{aligned}$$

$$\begin{aligned}
(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) &\to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}\\
(4)\ \mathsf{q}(x,0,\mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}\\
(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) &\to \{1 : 0\}^{\mathsf{false}}
\end{aligned}$$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\text{geo}} \arrow[d, swap, "\text{Chain Criterion}"] \\
\langle \mathcal{ADP}(\mathcal{R}_{\text{geo}}), \mathcal{ADP}(\mathcal{R}_{\text{geo}}), \varnothing \rangle \arrow[d, swap, "\mathrm{Proc}_{\mathtt{UR}}"] \arrow[d, "\mathrm{Pol}_0"] \\
\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle
\end{tikzcd}$$

$\mathcal{P}$-Dependency Graph

Dependency Graph Processor

$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}$
$(2)\ \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$

$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$
$(4)\ \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$
$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

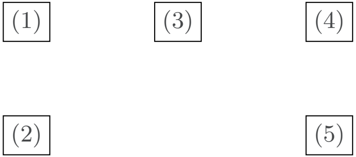
PTRS $\mathcal{R}_{\mathrm{geo}}$

Chain Criterion $\downarrow$

$\langle \mathcal{ADP}(\mathcal{R}_{\mathrm{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathrm{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathrm{UR}}$ $\downarrow$ Pol_0

$\langle \{(1), (2), (3), (4), (5)\}, \{(1), (2), (3), (4), (5)\}, \varnothing \rangle$



$\mathcal{P}$-Dependency Graph

Dependency Graph Processor

$$\begin{aligned}(1)\ \ \mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}\\ (2)\ \ \mathsf{geo}(x) &\to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}\end{aligned}$$

$$\begin{aligned}(3)\ \ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) &\to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}\\ (4)\ \ \mathsf{q}(x,0,\mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}\\ (5)\ \ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) &\to \{1 : 0\}^{\mathsf{false}}\end{aligned}$$

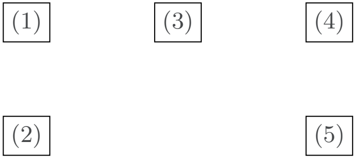
PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion $\downarrow$

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing\rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ $\downarrow$ Pol_0

$\langle\{(1),(2),(3),(4),(5)\},\{(1),(2),(3),(4),(5)\},\varnothing\rangle$



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$

Dependency Graph Processor

$$
\begin{aligned}
&(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}\\
&(2)\ \quad \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}
\end{aligned}
$$

$$
\begin{aligned}
&(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}\\
&(4)\ \quad \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}\\
&(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}
\end{aligned}
$$

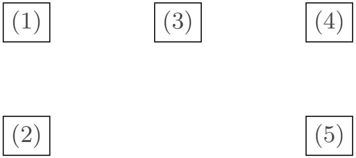
PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion $\downarrow$

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ $\downarrow$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$



(1) (3) (4)

(2) (5)

$\mathcal{P}$-Dependency Graph

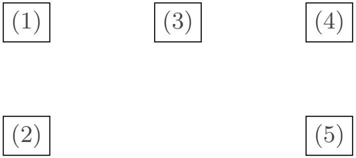
There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j

Dependency Graph Processor

$$\begin{aligned}
&(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}\\
&(2)\ \quad\ \mathsf{geo}(x) \to \{{}^1\!/\!{}_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/\!{}_2 : x\}^{\mathsf{true}}
\end{aligned}$$

$$\begin{aligned}
&(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}\\
&(4)\ \quad\ \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}\\
&(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}
\end{aligned}$$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathrm{UR}}"', "\mathrm{Pol}_0"] \\
\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle
\end{tikzcd}$$



$\mathcal{P}$-Dependency Graph

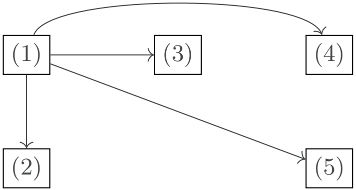
There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^{\#}$ for some r_j such that $t^{\#}\sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^{\#}\sigma_2$ for substitutions σ_1, σ_2.

Dependency Graph Processor

$(1)\ \mathsf{start}(x, y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}$
$(2)\ \quad \mathsf{geo}(x) \to \{{}^1\!/\!{}_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/\!{}_2 : x\}^{\mathsf{true}}$

$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$
$(4)\ \quad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$
$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathrm{UR}}"', "\mathrm{Pol}_0"] \\
\langle \{(1), (2), (3), (4), (5)\}, \{(1), (2), (3), (4), (5)\}, \varnothing \rangle
\end{tikzcd}$$



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^*_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

Dependency Graph Processor

$$\begin{aligned} &(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}} \\ &(2)\ \quad \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}} \end{aligned}$$

$$\begin{aligned} &(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}} \\ &(4)\ \quad \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}} \\ &(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}} \end{aligned}$$

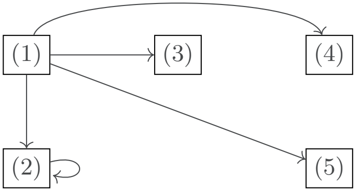
PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion $\downarrow$

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ $\downarrow$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$



$\mathcal{P}$-Dependency Graph

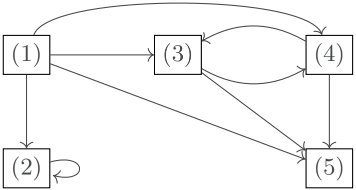
There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

Dependency Graph Processor

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}$$
$$(2)\ \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}$$
$$(4)\ \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}$$
$$(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$$

$$\begin{tikzcd} \text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\ \langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{UR}}"', "\mathrm{Pol}_0"] \\ \langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle \end{tikzcd}$$



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\#\sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^\#\sigma_2$ for substitutions σ_1, σ_2.

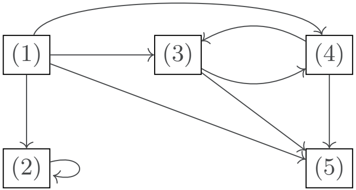
Dependency Graph Processor

$$
\begin{array}{ll}
(1) & \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}} \\
(2) & \mathsf{geo}(x) \to \{{}^1\!/_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/_2 : x\}^{\mathsf{true}}
\end{array}
$$

$$
\begin{array}{ll}
(3) & \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}} \\
(4) & \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}} \\
(5) & \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}
\end{array}
$$

$$
\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{UR}}"', "\mathrm{Pol}_0"] \\
\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle
\end{tikzcd}
$$

$\mathrm{Proc}_{\mathtt{DG}}$ - "Consider each SCC + predecessors separately"



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^*_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

Dependency Graph Processor

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}$$
$$(2)\ \mathsf{geo}(x) \to \{{}^1\!/\!{}_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/\!{}_2 : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$$
$$(4)\ \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$$
$$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$$

PTRS $\mathcal{R}_{\mathsf{geo}}$

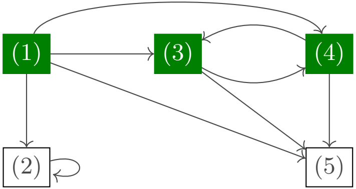
Chain Criterion ↓

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ ↓ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ - "Consider each SCC + predecessors separately"



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

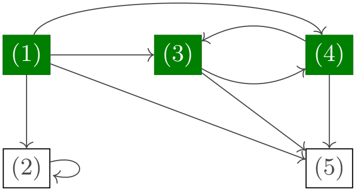
Dependency Graph Processor

$$\begin{aligned}
(1)\ \mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}\\
(2)\quad \mathsf{geo}(x) &\to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}
\end{aligned}$$

$$\begin{aligned}
(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) &\to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}\\
(4)\quad \mathsf{q}(x,0,\mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}\\
(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) &\to \{1 : 0\}^{\mathsf{false}}
\end{aligned}$$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{UR}}"', "\mathrm{Pol}_0"] \\
\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle
\end{tikzcd}$$

$\mathrm{Proc}_{\mathtt{DG}}$ - “Consider each SCC + predecessors separately”



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

Dependency Graph Processor

$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\ \ (x), y, y)\}^{\mathsf{false}}$
$(2)\qquad \mathsf{geo}(x) \to \{{}^{1}/_{2} : \mathsf{geo}\ \ (\mathsf{s}(x)),\ {}^{1}/_{2} : x\}^{\mathsf{true}}$

$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$
$(4)\qquad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$
$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

PTRS $\mathcal{R}_{\mathsf{geo}}$

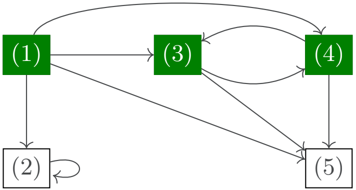
Chain Criterion $\downarrow$

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ $\downarrow$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ - "Consider each SCC + predecessors separately"



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

Dependency Graph Processor

$$\begin{aligned}
(1)\quad \mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\ (x), y, y)\}^{\mathsf{false}}\\
(2)\quad \mathsf{geo}(x) &\to \{1/2 : \mathsf{geo}\ (\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}
\end{aligned}$$

$$\begin{aligned}
(3)\quad \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) &\to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}\\
(4)\quad \mathsf{q}(x, 0, \mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}\\
(5)\quad \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) &\to \{1 : 0\}^{\mathsf{false}}
\end{aligned}$$

PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion $\downarrow$

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

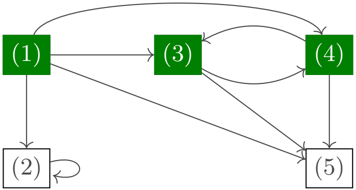
$\mathrm{Proc}_{\mathtt{UR}}$ $\downarrow$ Pol_0

$\langle \{(1), (2), (3), (4), (5)\}, \{(1), (2), (3), (4), (5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ $\downarrow$ Pol_0

$\langle \mathsf{q\text{-}SubProblem}, \mathsf{q\text{-}SubProblem}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ - "Consider each SCC + predecessors separately"



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

Dependency Graph Processor

$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}$
$(2)\ \qquad \mathsf{geo}(x) \to \{{}^1\!/\!_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/\!_2 : x\}^{\mathsf{true}}$

$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$
$(4)\ \qquad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$
$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion $\downarrow$

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

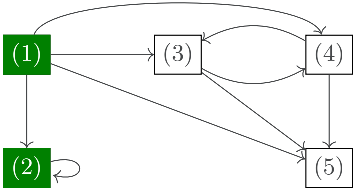
$\mathrm{Proc}_{\mathtt{UR}}$ $\downarrow$ Pol_0

$\langle \{(1), (2), (3), (4), (5)\}, \{(1), (2), (3), (4), (5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ $\downarrow$ Pol_0

$\langle$q-SubProblem, q-SubProblem, $\varnothing\rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ - "Consider each SCC + predecessors separately"



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

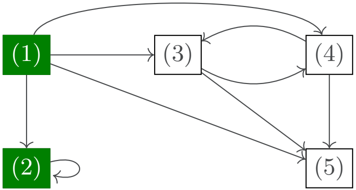
Dependency Graph Processor

$$\begin{array}{ll}(1) & \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}} \\ (2) & \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}\end{array}$$

$$\begin{array}{ll}(3) & \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}} \\ (4) & \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}} \\ (5) & \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}\end{array}$$

$$\begin{tikzcd} \text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\ \langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{UR}}"', "\mathrm{Pol}_0"] \\ \langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{DG}}"', "\mathrm{Pol}_0"] \\ \langle \text{q-SubProblem}, \text{q-SubProblem}, \varnothing \rangle \end{tikzcd}$$

$\mathrm{Proc}_{\mathtt{DG}}$ - "Consider each SCC + predecessors separately"



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

Dependency Graph Processor

$$\begin{array}{ll}(1) & \mathsf{start}(x,y) \to \{1 : \mathsf{q}\;\,(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}} \\ (2) & \qquad \mathsf{geo}(x) \to \{{}^1\!/\!{}_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/\!{}_2 : x\}^{\mathsf{true}}\end{array}$$

$$\begin{array}{ll}(3) & \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}\;\,(x, y, z)\}^{\mathsf{false}} \\ (4) & \qquad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}\;\,(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}} \\ (5) & \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}\end{array}$$

PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion $\downarrow$

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

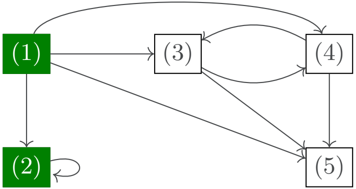
$\mathrm{Proc}_{\mathtt{UR}}$ $\downarrow$ Pol_0

$\langle \{(1), (2), (3), (4), (5)\}, \{(1), (2), (3), (4), (5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ $\downarrow$ Pol_0

$\langle \mathsf{q\text{-}SubProblem}, \mathsf{q\text{-}SubProblem}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ - “Consider each SCC + predecessors separately”



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

Dependency Graph Processor

$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}\ \ (\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}$
$(2)\qquad \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$

$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}\ \ (x, y, z)\}^{\mathsf{false}}$
$(4)\qquad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}\ \ (x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$
$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion ↓

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

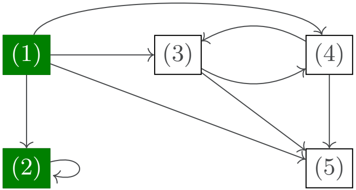
$\mathrm{Proc}_{\mathtt{UR}}$ ↓ Pol_0

$\langle \{(1), (2), (3), (4), (5)\}, \{(1), (2), (3), (4), (5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ ↓ Pol_0 → $\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \mathsf{q\text{-}SubProblem}, \mathsf{q\text{-}SubProblem}, \varnothing \rangle$ $\quad$ $\langle \mathsf{geo\text{-}SubProblem}, \mathsf{geo\text{-}SubProblem}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ - "Consider each SCC + predecessors separately"



$\mathcal{P}$-Dependency Graph

There is an arc from $\ldots \to \{p_1 : r_1, \ldots, p_k : r_k\}$ to $v \to \ldots$ if there is a subterm $t^\#$ for some r_j such that $t^\# \sigma_1 \xrightarrow{\mathsf{i}}{}^{*}_{\mathsf{np}(\mathcal{P})} v^\# \sigma_2$ for substitutions σ_1, σ_2.

Reduction Pair Processor

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\ (x), y, y)\}^{\mathsf{false}}$$
$$(2)\ \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}\ (\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$$
$$(4)\ \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$$
$$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

Reduction Pair Processor

(1) $\mathsf{start}(x, y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\ \ (x), y, y)\}^{\mathsf{false}}$
(2) $\mathsf{geo}(x) \to \{1/2 : \mathsf{geo}\ \ (\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$

(3) $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$
(4) $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$
(5) $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$

Reduction Pair Processor

$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\ (x), y, y)\}^{\mathsf{false}}$
$(2)\quad \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}\ (\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$

$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$
$(4)\quad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$
$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P} : \mathcal{I}(\ell^\#) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$

Reduction Pair Processor

$$\begin{aligned}(1)\ \mathsf{start}(x,y) &\to \{1 : \mathsf{q}^{\sharp}(\mathsf{geo}\ \ (x), y, y)\}^{\mathsf{false}}\\ (2)\ \ \ \ \mathsf{geo}(x) &\to \{1/2 : \mathsf{geo}\ \ (\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}\end{aligned}$$

$$\begin{aligned}(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) &\to \{1 : \mathsf{q}^{\sharp}(x,y,z)\}^{\mathsf{false}}\\ (4)\ \ \ \ \mathsf{q}(x,0,\mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}^{\sharp}(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}\\ (5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) &\to \{1 : 0\}^{\mathsf{false}}\end{aligned}$$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P} : \mathcal{I}(\ell^{\#}) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_{\#} r_j} \mathcal{I}(t^{\#})$

Reduction Pair Processor

$(1)\ \mathsf{start}(x, y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\ (x), y, y)\}^{\mathsf{false}}$

$(2)\ \quad \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}\ (\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$

$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$

$(4)\ \quad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$

$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P} : \mathcal{I}(\ell^\#) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$

Reduction Pair Processor

$$
\begin{aligned}
&(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\ \ (x), y, y)\}^{\mathsf{false}}\\
&(2)\qquad \mathsf{geo}(x) \to \{{}^1\!/{}_2 : \mathsf{geo}\ \ (\mathsf{s}(x)),\ {}^1\!/{}_2 : x\}^{\mathsf{true}}
\end{aligned}
$$

$$
\begin{aligned}
&(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}\\
&(4)\qquad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}\\
&(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}
\end{aligned}
$$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P} : \mathcal{I}(\ell^\#) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P}_> : \mathcal{I}(\ell^\#) > \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$

Reduction Pair Processor

$$
\begin{array}{ll}
(1) & \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\;\,(x), y, y)\}^{\mathsf{false}} \\
(2) & \qquad \mathsf{geo}(x) \to \{{}^{1}\!/{}_{2} : \mathsf{geo}\;\,(\mathsf{s}(x)),\ {}^{1}\!/{}_{2} : x\}^{\mathsf{true}}
\end{array}
$$

$$
\begin{array}{ll}
(3) & \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}} \\
(4) & \qquad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}} \\
(5) & \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}
\end{array}
$$

$$
\begin{array}{rr}
\mathcal{I}_0 = 0 & \mathcal{I}_{\mathsf{s}}(x) = x + 1 \\
\mathcal{I}_{\mathsf{geo}}(x) = x + 1 & \mathcal{I}_{\mathsf{geo}^\sharp}(x) = 1 \\
\mathcal{I}_{\mathsf{q}}(x, y, z) = x + 1 & \mathcal{I}_{\mathsf{q}^\sharp}(x, y, z) = x + 1 \\
\mathcal{I}_{\mathsf{start}}(x, y) = x + 3 &
\end{array}
$$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P} : \mathcal{I}(\ell^{\#}) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_{\#} r_j} \mathcal{I}(t^{\#})$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P}_{>} : \mathcal{I}(\ell^{\#}) > \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_{\#} r_j} \mathcal{I}(t^{\#})$

Reduction Pair Processor

$$
\begin{array}{rrl}
(1) & \mathsf{start}(x,y) & \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\;\,(x), y, y)\}^{\mathsf{false}} \\
(2) & \mathsf{geo}(x) & \to \{1/2 : \mathsf{geo}\;\,(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}
\end{array}
$$

$$
\begin{array}{rrl}
(3) & \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) & \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}} \\
(4) & \mathsf{q}(x, 0, \mathsf{s}(z)) & \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}} \\
(5) & \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) & \to \{1 : 0\}^{\mathsf{false}}
\end{array}
$$

$$
\begin{array}{rlrl}
\mathcal{I}_0 & = 0 & \mathcal{I}_{\mathsf{s}}(x) & = x + 1 \\
\mathcal{I}_{\mathsf{geo}}(x) & = x + 1 & \mathcal{I}_{\mathsf{geo}^\sharp}(x) & = 1 \\
\mathcal{I}_{\mathsf{q}}(x, y, z) & = x + 1 & \mathcal{I}_{\mathsf{q}^\sharp}(x, y, z) & = x + 1 \\
\mathcal{I}_{\mathsf{start}}(x, y) & = x + 3 & &
\end{array}
$$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^m \in \mathcal{P} : \mathcal{I}(\ell^\#) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^m \in \mathcal{P}_> : \mathcal{I}(\ell^\#) > \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$

Reduction Pair Processor

$(1)\ \mathsf{start}(x, y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\ \ (x), y, y)\}^{\mathsf{false}}$

$(2)\qquad x + 1 \geq {}^1\!/_2 \cdot x + 2 + {}^1\!/_2 \cdot 0$

$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$

$(4)\qquad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$

$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

$$\begin{aligned} \mathcal{I}_0 &= 0 & \mathcal{I}_{\mathsf{s}}(x) &= x + 1 \\ \mathcal{I}_{\mathsf{geo}}(x) &= x + 1 & \mathcal{I}_{\mathsf{geo}^\sharp}(x) &= 1 \\ \mathcal{I}_{\mathsf{q}}(x, y, z) &= x + 1 & \mathcal{I}_{\mathsf{q}^\sharp}(x, y, z) &= x + 1 \\ \mathcal{I}_{\mathsf{start}}(x, y) &= x + 3 \end{aligned}$$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^m \in \mathcal{P} : \mathcal{I}(\ell^\#) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^m \in \mathcal{P}_> : \mathcal{I}(\ell^\#) > \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$

Reduction Pair Processor

$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\ \ (x), y, y)\}^{\mathsf{false}}$

$(2)\qquad x+1 \geq x+1$

$(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}$

$(4)\qquad \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}$

$(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

$$\begin{aligned} \mathcal{I}_0 &= 0 & \mathcal{I}_{\mathsf{s}}(x) &= x+1 \\ \mathcal{I}_{\mathsf{geo}}(x) &= x+1 & \mathcal{I}_{\mathsf{geo}^\sharp}(x) &= 1 \\ \mathcal{I}_{\mathsf{q}}(x,y,z) &= x+1 & \mathcal{I}_{\mathsf{q}^\sharp}(x,y,z) &= x+1 \\ \mathcal{I}_{\mathsf{start}}(x,y) &= x+3 \end{aligned}$$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P} : \mathcal{I}(\ell^\#) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P}_> : \mathcal{I}(\ell^\#) > \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$

Reduction Pair Processor

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}\ \ (x), y, y)\}^{\mathsf{false}}$$
$$(2)\qquad \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}\ \ (\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$$
$$(4)\qquad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$$
$$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$$

$$\begin{aligned} \mathcal{I}_0 &= 0 & \mathcal{I}_{\mathsf{s}}(x) &= x+1 \\ \mathcal{I}_{\mathsf{geo}}(x) &= x+1 & \mathcal{I}_{\mathsf{geo}^\sharp}(x) &= 1 \\ \mathcal{I}_{\mathsf{q}}(x,y,z) &= x+1 & \mathcal{I}_{\mathsf{q}^\sharp}(x,y,z) &= x+1 \\ \mathcal{I}_{\mathsf{start}}(x,y) &= x+3 \end{aligned}$$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^m \in \mathcal{P} : \mathcal{I}(\ell^\#) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^m \in \mathcal{P}_> : \mathcal{I}(\ell^\#) > \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$

Reduction Pair Processor

$$\begin{array}{ll}(1) & x+3 > x+2\\ (2) & 1 > 0\end{array}$$

$$\begin{array}{ll}(3) & x+2 > x+1\\ (4) & x+1 \geq x+1\\ (5) & 1 > 0\end{array}$$

$$\begin{aligned}\mathcal{I}_{0} &= 0 & \mathcal{I}_{\mathsf{s}}(x) &= x+1\\ \mathcal{I}_{\mathsf{geo}}(x) &= x+1 & \mathcal{I}_{\mathsf{geo}^\sharp}(x) &= 1\\ \mathcal{I}_{\mathsf{q}}(x,y,z) &= x+1 & \mathcal{I}_{\mathsf{q}^\sharp}(x,y,z) &= x+1\\ \mathcal{I}_{\mathsf{start}}(x,y) &= x+3 & &\end{aligned}$$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P} : \mathcal{I}(\ell^\#) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P}_> : \mathcal{I}(\ell^\#) > \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$

Reduction Pair Processor

$$\begin{aligned}(1)\ & x+3 > x+2\\ (2)\ & \quad\ 1 > 0\end{aligned}$$

$$\text{Proc}_{\text{DG}} \ \big\downarrow\ \text{Pol}_0$$

$$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$$

$$\begin{aligned}(3)\ & x+2 > x+1\\ (4)\ & x+1 \geq x+1\\ (5)\ & \quad\ 1 > 0\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_0 &= 0 & \mathcal{I}_{\mathsf{s}}(x) &= x+1\\
\mathcal{I}_{\mathsf{geo}}(x) &= x+1 & \mathcal{I}_{\mathsf{geo}^\sharp}(x) &= 1\\
\mathcal{I}_{\mathsf{q}}(x,y,z) &= x+1 & \mathcal{I}_{\mathsf{q}^\sharp}(x,y,z) &= x+1\\
\mathcal{I}_{\mathsf{start}}(x,y) &= x+3
\end{aligned}$$

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P} : \mathcal{I}(\ell^{\#}) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_{\#} r_j} \mathcal{I}(t^{\#})$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P}_{>} : \mathcal{I}(\ell^{\#}) > \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_{\#} r_j} \mathcal{I}(t^{\#})$

Reduction Pair Processor

$$\begin{array}{ll} (1) & x+3 > x+2 \\ (2) & \quad 1 > 0 \end{array}$$

$\mathrm{Proc}_{\mathtt{DG}} \downarrow \mathrm{Pol}_0$

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$$\begin{array}{ll} (3) & x+2 > x+1 \\ (4) & x+1 \geq x+1 \\ (5) & \quad 1 > 0 \end{array}$$

$$\begin{array}{rlrl} \mathcal{I}_0 &= 0 & \mathcal{I}_{\mathsf{s}}(x) &= x+1 \\ \mathcal{I}_{\mathsf{geo}}(x) &= x+1 & \mathcal{I}_{\mathsf{geo}^\sharp}(x) &= 1 \\ \mathcal{I}_{\mathsf{q}}(x,y,z) &= x+1 & \mathcal{I}_{\mathsf{q}^\sharp}(x,y,z) &= x+1 \\ \mathcal{I}_{\mathsf{start}}(x,y) &= x+3 & & \end{array}$$

$\mathrm{Proc}_{\mathtt{RP}}$ - “Highest degree gives upper bound on complexity of rules in $\mathcal{P}_>$”

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P} : \mathcal{I}(\ell^\#) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P}_> : \mathcal{I}(\ell^\#) > \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$

Reduction Pair Processor

$(1)\ x + 3 > x + 2$
$(2)\ \quad 1 > 0$

$(3)\ x + 2 > x + 1$
$(4)\ x + 1 \geq x + 1$
$(5)\ \quad 1 > 0$

$$\mathrm{Proc}_{\mathtt{DG}} \quad \big\downarrow \quad \mathrm{Pol}_0$$

$$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$$

$$\mathrm{Proc}_{\mathtt{RP}} \quad \big\downarrow \quad \mathrm{Pol}_1$$

$$\langle \{(1),(2),(3),(4),(5)\}, \{(4)\}, \{(1),(2),(3),(5)\} \rangle$$

$$\mathcal{I}_0 = 0 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1$$

$$\mathcal{I}_{\mathsf{geo}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{geo}^\sharp}(x) = 1$$

$$\mathcal{I}_{\mathsf{q}}(x, y, z) = x + 1 \qquad \mathcal{I}_{\mathsf{q}^\sharp}(x, y, z) = x + 1$$

$$\mathcal{I}_{\mathsf{start}}(x, y) = x + 3$$

$\mathrm{Proc}_{\mathtt{RP}}$ - “Highest degree gives upper bound on complexity of rules in $\mathcal{P}_>$”

Weakly Monotonic, Multilinear, CPI $\mathcal{I}$:

- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{\mathsf{true}} \in \mathcal{P} : \mathcal{I}(\ell) \geq \sum_{1 \leq j \leq k} p_j \cdot \mathcal{I}(\flat(r_j))$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P} : \mathcal{I}(\ell^\#) \geq \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$
- For every $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\}^{m} \in \mathcal{P}_> : \mathcal{I}(\ell^\#) > \sum_{1 \leq j \leq k} p_j \cdot \sum_{t \trianglelefteq_\# r_j} \mathcal{I}(t^\#)$

Knowledge Propagation Processor

$$\begin{aligned}(1)\ \mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}\\ (2)\ \mathsf{geo}(x) &\to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}\end{aligned}$$

$$\begin{aligned}(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) &\to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}\\ (4)\ \mathsf{q}(x,0,\mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}\\ (5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) &\to \{1 : 0\}^{\mathsf{false}}\end{aligned}$$

Knowledge Propagation Processor

$$
\begin{array}{ll}
(1) & \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}} \\
(2) & \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}
\end{array}
$$

$$
\begin{array}{ll}
(3) & \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}} \\
(4) & \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}} \\
(5) & \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}
\end{array}
$$

$$
\begin{tikzcd}
\ldots \arrow[d, "\mathrm{Proc}_{\mathrm{RP}}"', "\mathrm{Pol}_1"] \\
\langle\{(1),(2),(3),(4),(5)\},\{(4)\},\{(1),(2),(3),(5)\}\rangle
\end{tikzcd}
$$

Knowledge Propagation Processor

$$\begin{array}{ll}(1) & \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}} \\ (2) & \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}\end{array}$$

$$\begin{array}{ll}(3) & \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}} \\ (4) & \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}} \\ (5) & \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}\end{array}$$

$$\begin{tikzcd} \dots \arrow[d, "\mathrm{Proc}_{\mathrm{RP}}"', "\mathrm{Pol}_1"] \\ \langle\{(1),(2),(3),(4),(5)\},\{(4)\},\{(1),(2),(3),(5)\}\rangle \end{tikzcd}$$

Knowledge Propagation - $\mathrm{Proc}_{\mathrm{KP}}$

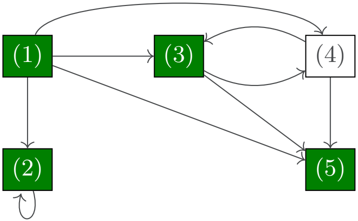
If all predecessors of a node $s \to \mu$ in the dependency graph have known complexity, then the complexity of $s \to \mu$ is already accounted for.

Knowledge Propagation Processor

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}$$
$$(2)\quad \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$$
$$(4)\quad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$$
$$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$$

$$\begin{tikzcd} \cdots \arrow[d, "\mathrm{Proc}_{\mathrm{RP}}"', "\mathrm{Pol}_1"] \\ \langle \{(1),(2),(3),(4),(5)\}, \{(4)\}, \{(1),(2),(3),(5)\} \rangle \end{tikzcd}$$



Knowledge Propagation - $\mathrm{Proc}_{\mathrm{KP}}$

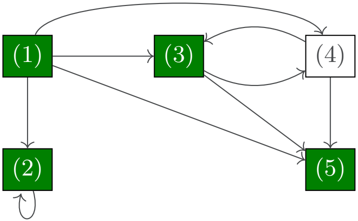
If all predecessors of a node $s \to \mu$ in the dependency graph have known complexity, then the complexity of $s \to \mu$ is already accounted for.

Knowledge Propagation Processor

$$\begin{array}{ll}(1) & \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}} \\ (2) & \mathsf{geo}(x) \to \{{}^1\!/_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/_2 : x\}^{\mathsf{true}}\end{array}$$

$$\begin{array}{ll}(3) & \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}} \\ (4) & \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}} \\ (5) & \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}\end{array}$$

$$\begin{tikzcd}
\ldots \arrow[d, "\mathrm{Proc_{RP}}"', "\mathrm{Pol}_1"] \\
\langle\{(1),(2),(3),(4),(5)\},\{(4)\},\{(1),(2),(3),(5)\}\rangle \arrow[d, "\mathrm{Proc_{KP}}"', "\mathrm{Pol}_0"] \\
\langle\{(1),(2),(3),(4),(5)\},\varnothing,\{(1),(2),(3),(4),(5)\}\rangle
\end{tikzcd}$$



Knowledge Propagation - $\mathrm{Proc_{KP}}$

If all predecessors of a node $s \to \mu$ in the dependency graph have known complexity, then the complexity of $s \to \mu$ is already accounted for.

Final Expected Complexity Proof

$$
\begin{aligned}
&(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}\\
&(2)\qquad \mathsf{geo}(x) \to \{{}^1\!/\!_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/\!_2 : x\}^{\mathsf{true}}
\end{aligned}
$$

$$
\begin{aligned}
&(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}\\
&(4)\qquad \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}\\
&(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}
\end{aligned}
$$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), {\color{orange}\mathcal{ADP}(\mathcal{R}_{\mathsf{geo}})}, \varnothing \rangle
\end{tikzcd}$$

Final Expected Complexity Proof

$$
\begin{aligned}
&(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}\\
&(2)\ \quad \mathsf{geo}(x) \to \{{}^1\!/\!{}_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/\!{}_2 : x\}^{\mathsf{true}}
\end{aligned}
$$

$$
\begin{aligned}
&(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}\\
&(4)\ \quad \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}\\
&(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}
\end{aligned}
$$

PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion $\downarrow$

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ $\downarrow$ Pol_0

$\langle \{(1), (2), (3), (4), (5)\}, \{(1), (2), (3), (4), (5)\}, \varnothing \rangle$

Final Expected Complexity Proof

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}$$

$$(2)\ \mathsf{geo}(x) \to \{{}^1\!/\!{}_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/\!{}_2 : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}$$

$$(4)\ \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}$$

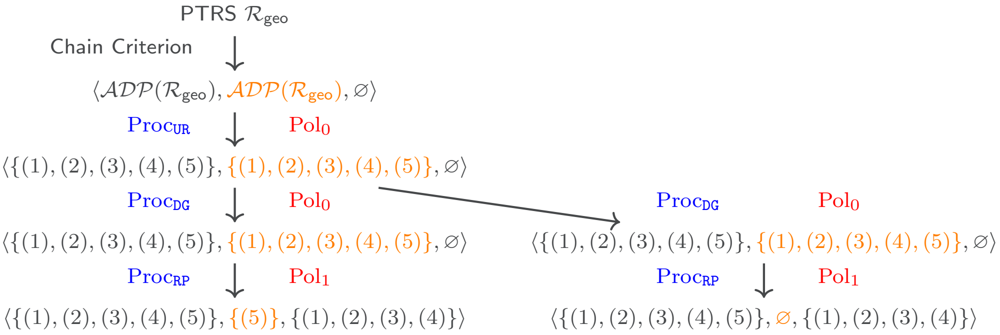
$$(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$$

$$\begin{tikzcd}
\mathsf{PTRS}\ \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] & \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{UR}}"', "\mathrm{Pol}_0"] & \\
\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{DG}}"', "\mathrm{Pol}_0"] \arrow[dr, "\mathrm{Proc}_{\mathtt{DG}}\quad \mathrm{Pol}_0"] & \\
\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle & \langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle
\end{tikzcd}$$

Final Expected Complexity Proof

$$
\begin{array}{ll}
(1) & \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}} \\
(2) & \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}
\end{array}
$$

$$
\begin{array}{ll}
(3) & \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}} \\
(4) & \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}} \\
(5) & \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}
\end{array}
$$



PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion ↓

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ ↓ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ ↓ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ ↓ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \{(5)\}, \{(1),(2),(3),(4)\} \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ ↓ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4)\} \rangle$

Final Expected Complexity Proof

(1) $\mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}$
(2) $\mathsf{geo}(x) \to \{{}^1\!/_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/_2 : x\}^{\mathsf{true}}$

(3) $\mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}$
(4) $\mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}$
(5) $\mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$

$$\begin{tikzcd}
\text{PTRS } \mathcal{R}_{\mathsf{geo}} \arrow[d, "\text{Chain Criterion}"'] & \\
\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{UR}}"', "\mathrm{Pol}_0"] & \\
\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{DG}}"', "\mathrm{Pol}_0"] \arrow[dr, "\mathrm{Proc}_{\mathtt{DG}}"', "\mathrm{Pol}_0"] & \\
\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{RP}}"', "\mathrm{Pol}_1"] & \langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{RP}}"', "\mathrm{Pol}_1"] \\
\langle \{(1),(2),(3),(4),(5)\}, \{(5)\}, \{(1),(2),(3),(4)\} \rangle \arrow[d, "\mathrm{Proc}_{\mathtt{KP}}"', "\mathrm{Pol}_0"] & \langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4)\} \rangle \\
\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4),(5)\} \rangle &
\end{tikzcd}$$

Final Expected Complexity Proof

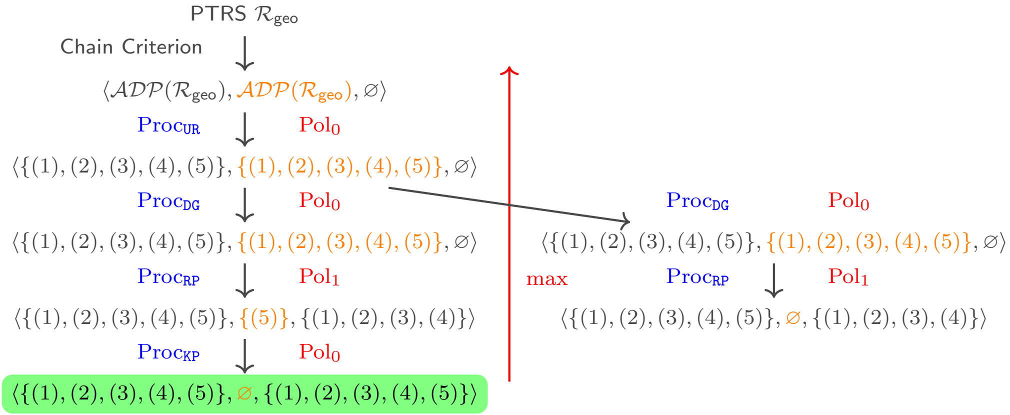
$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}$

$(2)\ \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$

$(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}$

$(4)\ \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}$

$(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$



PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \{(5)\}, \{(1),(2),(3),(4)\} \rangle$

$\mathrm{Proc}_{\mathtt{KP}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4),(5)\} \rangle$

max

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

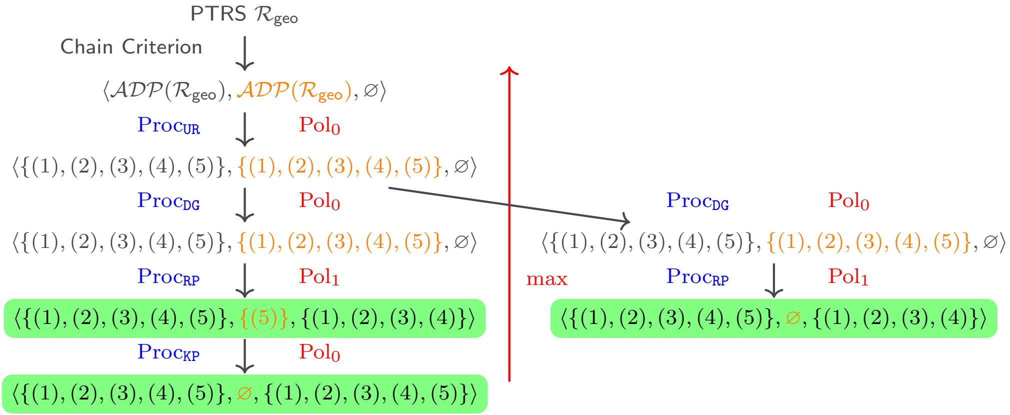
$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4)\} \rangle$

Final Expected Complexity Proof

$$\begin{aligned}(1)\ \ \mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}\\ (2)\ \ \mathsf{geo}(x) &\to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}\end{aligned}$$

$$\begin{aligned}(3)\ \ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) &\to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}\\ (4)\ \ \mathsf{q}(x,0,\mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}\\ (5)\ \ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) &\to \{1 : 0\}^{\mathsf{false}}\end{aligned}$$



PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion ↓

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ ↓ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ ↓ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ ↓ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \{(5)\}, \{(1),(2),(3),(4)\} \rangle$

$\mathrm{Proc}_{\mathtt{KP}}$ ↓ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4),(5)\} \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

max

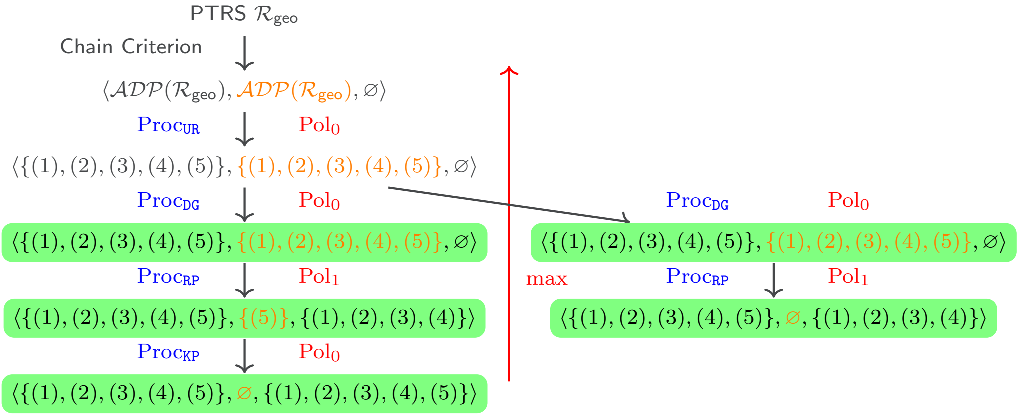
$\mathrm{Proc}_{\mathtt{RP}}$ ↓ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4)\} \rangle$

Final Expected Complexity Proof

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}$$
$$(2)\ \mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}$$
$$(4)\ \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}$$
$$(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$$



PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing\rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ Pol_0

$\langle\{(1),(2),(3),(4),(5)\},\{(1),(2),(3),(4),(5)\},\varnothing\rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle\{(1),(2),(3),(4),(5)\},\{(1),(2),(3),(4),(5)\},\varnothing\rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle\{(1),(2),(3),(4),(5)\},\{(5)\},\{(1),(2),(3),(4)\}\rangle$

$\mathrm{Proc}_{\mathtt{KP}}$ Pol_0

$\langle\{(1),(2),(3),(4),(5)\},\varnothing,\{(1),(2),(3),(4),(5)\}\rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle\{(1),(2),(3),(4),(5)\},\{(1),(2),(3),(4),(5)\},\varnothing\rangle$

max

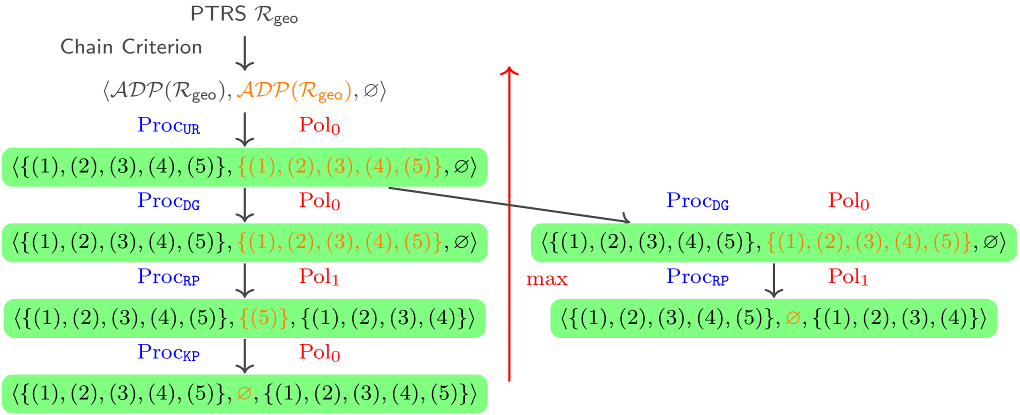
$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle\{(1),(2),(3),(4),(5)\},\varnothing,\{(1),(2),(3),(4)\}\rangle$

Final Expected Complexity Proof

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}$$
$$(2)\ \mathsf{geo}(x) \to \{\tfrac{1}{2} : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ \tfrac{1}{2} : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}$$
$$(4)\ \mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}$$
$$(5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$$



PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \{(5)\}, \{(1),(2),(3),(4)\} \rangle$

$\mathrm{Proc}_{\mathtt{KP}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4),(5)\} \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

max

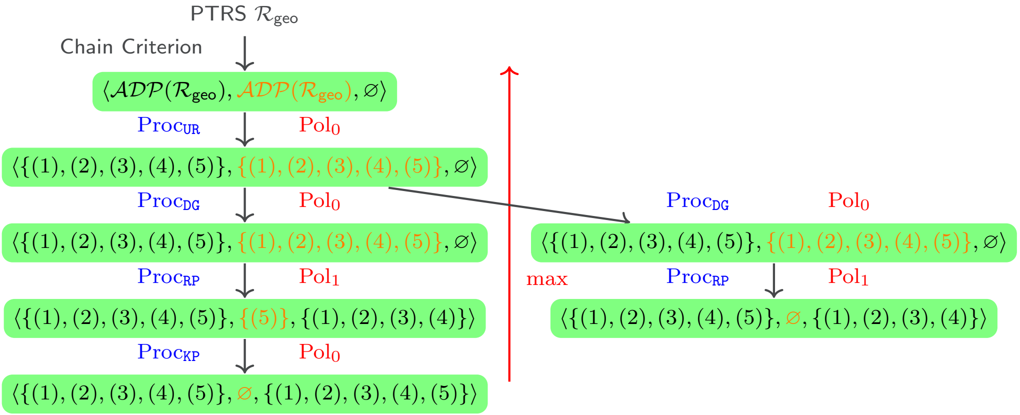
$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4)\} \rangle$

Final Expected Complexity Proof

$$(1)\ \mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}$$
$$(2)\ \mathsf{geo}(x) \to \{\nicefrac{1}{2} : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ \nicefrac{1}{2} : x\}^{\mathsf{true}}$$

$$(3)\ \mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$$
$$(4)\ \mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$$
$$(5)\ \mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$$



PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \{(5)\}, \{(1),(2),(3),(4)\} \rangle$

$\mathrm{Proc}_{\mathtt{KP}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4),(5)\} \rangle$

max

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4)\} \rangle$

Final Expected Complexity Proof

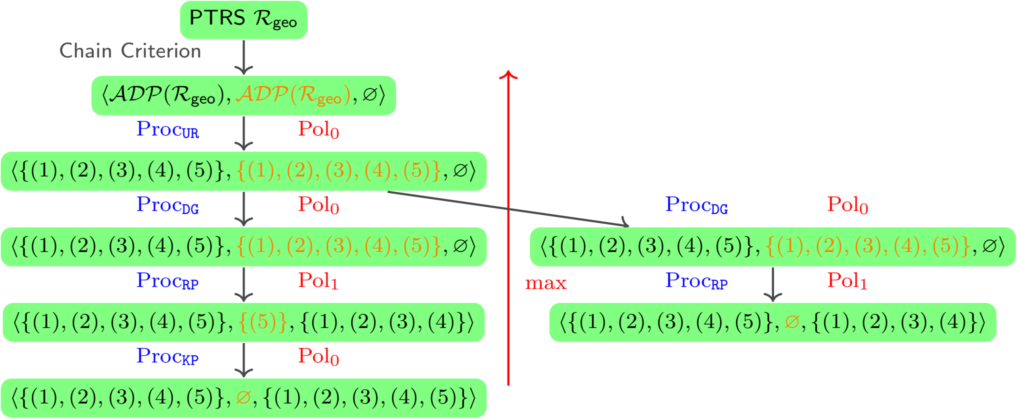
(1) $\mathsf{start}(x, y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}$

(2) $\mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$

(3) $\mathsf{q}(\mathsf{s}(x), \mathsf{s}(y), z) \to \{1 : \mathsf{q}^\sharp(x, y, z)\}^{\mathsf{false}}$

(4) $\mathsf{q}(x, 0, \mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x, \mathsf{s}(z), \mathsf{s}(z)))\}^{\mathsf{false}}$

(5) $\mathsf{q}(0, \mathsf{s}(y), \mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$



PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \{(5)\}, \{(1),(2),(3),(4)\} \rangle$

$\mathrm{Proc}_{\mathtt{KP}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4),(5)\} \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

max

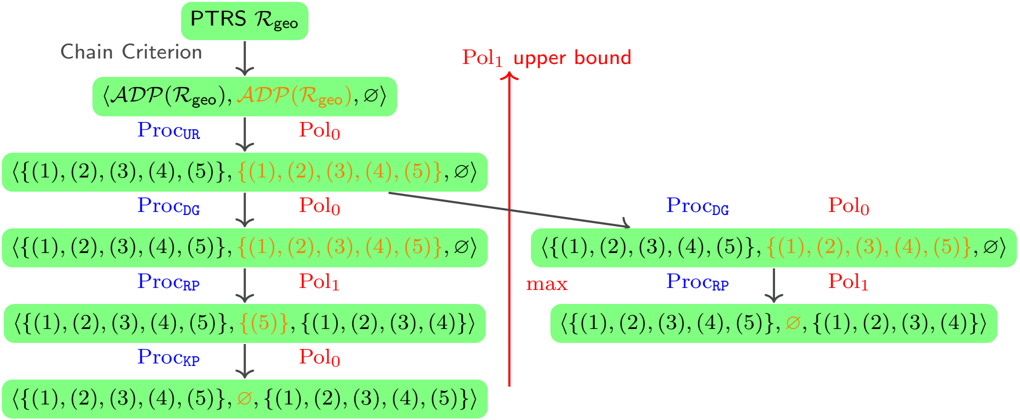
$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4)\} \rangle$

Final Expected Complexity Proof

$$\begin{aligned}(1)\ \mathsf{start}(x,y) &\to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x), y, y)\}^{\mathsf{false}}\\ (2)\quad \mathsf{geo}(x) &\to \{{}^1\!/_2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ {}^1\!/_2 : x\}^{\mathsf{true}}\end{aligned}$$

$$\begin{aligned}(3)\ \mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) &\to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}\\ (4)\quad \mathsf{q}(x,0,\mathsf{s}(z)) &\to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}\\ (5)\ \mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) &\to \{1 : 0\}^{\mathsf{false}}\end{aligned}$$



PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing\rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ Pol_0

$\langle\{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing\rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle\{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing\rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle\{(1),(2),(3),(4),(5)\}, \{(5)\}, \{(1),(2),(3),(4)\}\rangle$

$\mathrm{Proc}_{\mathtt{KP}}$ Pol_0

$\langle\{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4),(5)\}\rangle$

Pol_1 upper bound

max

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle\{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing\rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle\{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4)\}\rangle$

Final Expected Complexity Proof

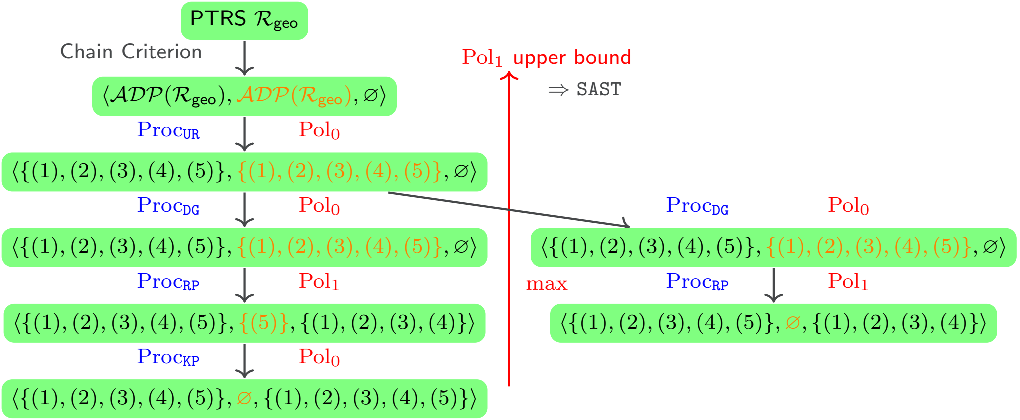
(1) $\mathsf{start}(x,y) \to \{1 : \mathsf{q}^\sharp(\mathsf{geo}^\sharp(x),y,y)\}^{\mathsf{false}}$

(2) $\mathsf{geo}(x) \to \{1/2 : \mathsf{geo}^\sharp(\mathsf{s}(x)),\ 1/2 : x\}^{\mathsf{true}}$

(3) $\mathsf{q}(\mathsf{s}(x),\mathsf{s}(y),z) \to \{1 : \mathsf{q}^\sharp(x,y,z)\}^{\mathsf{false}}$

(4) $\mathsf{q}(x,0,\mathsf{s}(z)) \to \{1 : \mathsf{s}(\mathsf{q}^\sharp(x,\mathsf{s}(z),\mathsf{s}(z)))\}^{\mathsf{false}}$

(5) $\mathsf{q}(0,\mathsf{s}(y),\mathsf{s}(z)) \to \{1 : 0\}^{\mathsf{false}}$



PTRS $\mathcal{R}_{\mathsf{geo}}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \mathcal{ADP}(\mathcal{R}_{\mathsf{geo}}), \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{UR}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \{(5)\}, \{(1),(2),(3),(4)\} \rangle$

$\mathrm{Proc}_{\mathtt{KP}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4),(5)\} \rangle$

Pol_1 upper bound

$\Rightarrow$ SAST

max

$\mathrm{Proc}_{\mathtt{DG}}$ Pol_0

$\langle \{(1),(2),(3),(4),(5)\}, \{(1),(2),(3),(4),(5)\}, \varnothing \rangle$

$\mathrm{Proc}_{\mathtt{RP}}$ Pol_1

$\langle \{(1),(2),(3),(4),(5)\}, \varnothing, \{(1),(2),(3),(4)\} \rangle$

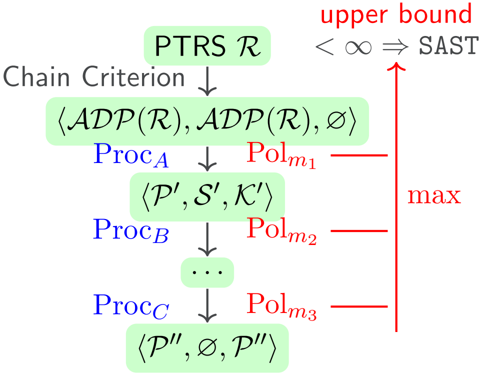
Implementation

- ADP framework for `SAST` analysis implemented in AProVE
- Evaluated on 138 PTRSs from TPDB (**T**ermination **P**roblem **D**ata**b**ase)
- Results on `SAST`:

Strategy	Start Terms	POLO	NaTT	AProVE
Full	Arbitrary	30	33	34
Full	Basic	30	33	43
Innermost	Arbitrary	30	33	45
Innermost	Basic	30	33	56

Conclusion

- ADP framework for expected complexity analysis (basic start terms + innermost rewriting)



upper bound

$< \infty \Rightarrow \texttt{SAST}$

PTRS $\mathcal{R}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle$

Proc_A Pol_{m_1}

$\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle$

Proc_B Pol_{m_2}

$\cdots$

Proc_C Pol_{m_3}

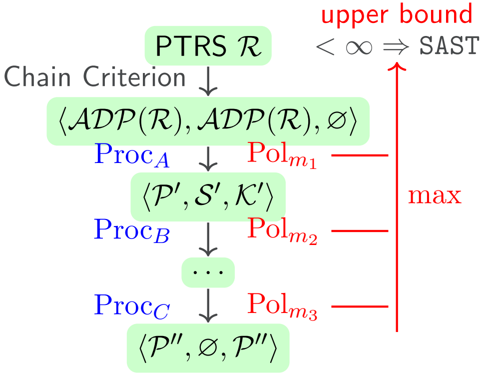
$\langle \mathcal{P}'', \varnothing, \mathcal{P}'' \rangle$

max

Conclusion

- ADP framework for expected complexity analysis (basic start terms + innermost rewriting)
- Adapted processors from existing DP framework

- Usable Rules Processor
- Dependency Graph Processor
- Reduction Pair Processor
- Knowledge Propagation Processor
- Probability Removal Processor



upper bound

PTRS $\mathcal{R}$ $< \infty \Rightarrow \mathtt{SAST}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle$

Proc_A Pol_{m_1}

$\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle$

Proc_B Pol_{m_2}

$\ldots$

Proc_C Pol_{m_3}

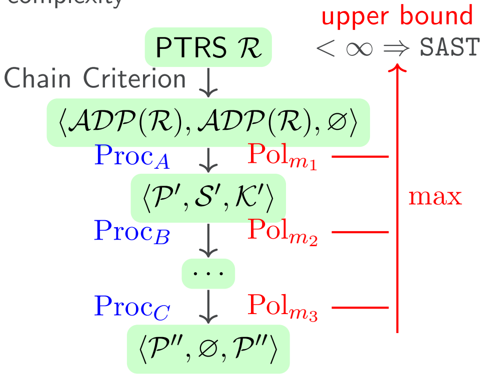
$\langle \mathcal{P}'', \varnothing, \mathcal{P}'' \rangle$

$\max$

Conclusion

- ADP framework for expected complexity analysis (basic start terms + innermost rewriting)
- Adapted processors from existing DP framework
- Future work:
    - Lift more processors to expected complexity

- Usable Rules Processor
- Dependency Graph Processor
- Reduction Pair Processor
- Knowledge Propagation Processor
- Probability Removal Processor



upper bound

PTRS $\mathcal{R}$

$< \infty \Rightarrow \mathtt{SAST}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle$

Proc_A Pol_{m_1}

$\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle$

max

Proc_B Pol_{m_2}

$\cdots$

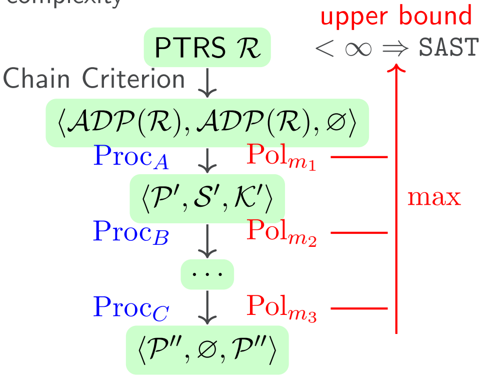
Proc_C Pol_{m_3}

$\langle \mathcal{P}'', \varnothing, \mathcal{P}'' \rangle$

Conclusion

- ADP framework for expected complexity analysis (basic start terms + innermost rewriting)
- Adapted processors from existing DP framework
- Future work:
  - Lift more processors to expected complexity
  - Integrate further reduction pairs

- Usable Rules Processor
- Dependency Graph Processor
- Reduction Pair Processor
- Knowledge Propagation Processor
- Probability Removal Processor



upper bound

PTRS $\mathcal{R}$

$< \infty \Rightarrow \mathtt{SAST}$

Chain Criterion

$\langle \mathcal{ADP}(\mathcal{R}), \mathcal{ADP}(\mathcal{R}), \varnothing \rangle$

Proc_A Pol_{m_1}

$\langle \mathcal{P}', \mathcal{S}', \mathcal{K}' \rangle$

max

Proc_B Pol_{m_2}

$\cdots$

Proc_C Pol_{m_3}

$\langle \mathcal{P}'', \varnothing, \mathcal{P}'' \rangle$