Modularity of Termination in Probabilistic Term Rewriting

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Termination of TRSs

$\mathcal{R}_{plus}$:

$$
\begin{aligned}
\mathsf{plus}(0, y) &\to y \\
\mathsf{plus}(\mathsf{s}(x), y) &\to \mathsf{s}(\mathsf{plus}(x, y))
\end{aligned}
$$

Termination of TRSs

$\mathcal{R}_{plus}$:

$$\begin{aligned} \mathsf{plus}(0, y) &\to y \\ \mathsf{plus}(\mathsf{s}(x), y) &\to \mathsf{s}(\mathsf{plus}(x, y)) \end{aligned}$$

$$\mathsf{plus}(\mathsf{s}(0), \mathsf{plus}(0, 0))$$

Termination of TRSs

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$\searrow$

$\mathrm{s}(\mathrm{plus}(0, \mathrm{plus}(0, 0)))$

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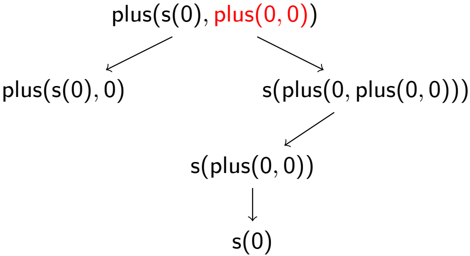
Termination of TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}(0, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$



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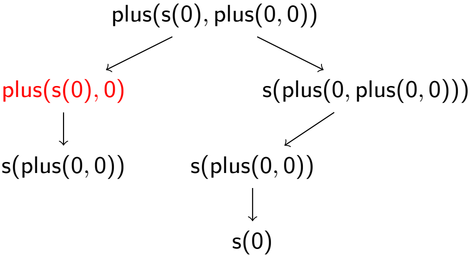

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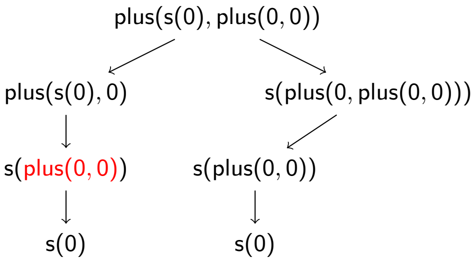

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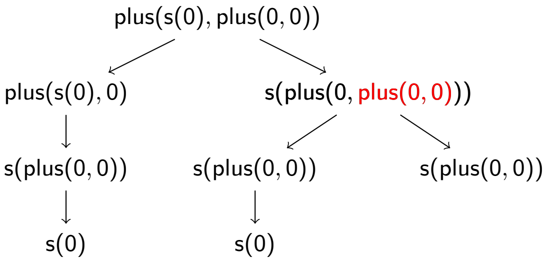

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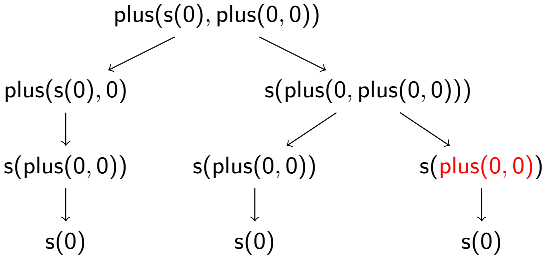

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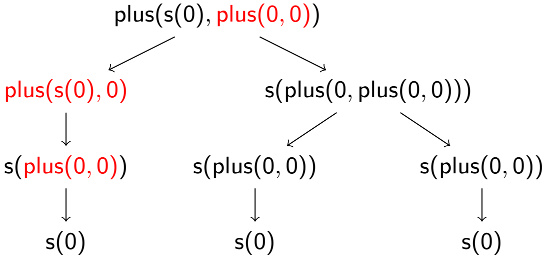

Innermost evaluation: always use an innermost reducible expression

Termination of TRSs

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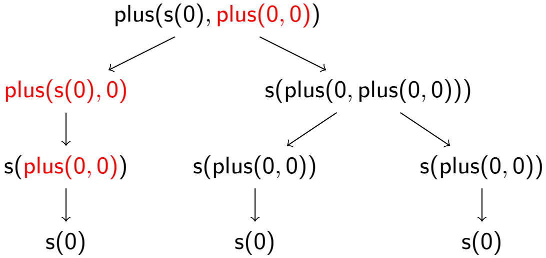

Innermost evaluation: always use an innermost reducible expression

Termination (Term)

$\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \ldots$

Modularity

Modularity

Imperative Programs:

Modularity

Imperative Programs:

$\mathcal{P}_1$ has property `Prop`
$\mathcal{P}_2$ has property `Prop`

Modularity

Imperative Programs:

$\mathcal{P}_1$ has property Prop
$\mathcal{P}_2$ has property Prop $\Longrightarrow$

Modularity

Imperative Programs:

$$\begin{matrix} \mathcal{P}_1 \text{ has property } \texttt{Prop} \\ \mathcal{P}_2 \text{ has property } \texttt{Prop} \end{matrix} \quad \Longrightarrow \quad \mathcal{P}_1;\mathcal{P}_2 \text{ has property } \texttt{Prop}$$

Modularity

Imperative Programs:

Sequential Execution

↑

$\mathcal{P}_1$ has property $\texttt{Prop}$
$\mathcal{P}_2$ has property $\texttt{Prop}$ $\Longrightarrow$ $\mathcal{P}_1; \mathcal{P}_2$ has property $\texttt{Prop}$

Modularity

Imperative Programs:

Sequential Execution

$\uparrow$

$\mathcal{P}_1$ has property Prop
$\mathcal{P}_2$ has property Prop $\implies$ $\mathcal{P}_1 ; \mathcal{P}_2$ has property Prop

Term Rewriting:

Modularity

Imperative Programs:

Sequential Execution

$\uparrow$

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$\mathcal{P}_2$ has property $\texttt{Prop}$ $\Longrightarrow$ $\mathcal{P}_1;\mathcal{P}_2$ has property $\texttt{Prop}$

Term Rewriting:

$\mathcal{R}_1$ has property $\texttt{Prop}$
$\mathcal{R}_2$ has property $\texttt{Prop}$

Modularity

Imperative Programs:

Sequential Execution

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$\mathcal{P}_2$ has property $\texttt{Prop}$ $\Longrightarrow$ $\mathcal{P}_1 ; \mathcal{P}_2$ has property $\texttt{Prop}$

Term Rewriting:

$\mathcal{R}_1$ has property $\texttt{Prop}$
$\mathcal{R}_2$ has property $\texttt{Prop}$ $\Longrightarrow$

Modularity

Imperative Programs:

Sequential Execution

↑

$\mathcal{P}_1$ has property Prop
$\mathcal{P}_2$ has property Prop $\Longrightarrow$ $\mathcal{P}_1 ; \mathcal{P}_2$ has property Prop

Term Rewriting:

$\mathcal{R}_1$ has property Prop
$\mathcal{R}_2$ has property Prop $\Longrightarrow$ $\mathcal{R}_1 \cup \mathcal{R}_2$ has property Prop

Modularity

Imperative Programs:

Sequential Execution ↑

$\mathcal{P}_1$ has property $\texttt{Prop}$
$\mathcal{P}_2$ has property $\texttt{Prop}$ $\implies$ $\mathcal{P}_1; \mathcal{P}_2$ has property $\texttt{Prop}$

Term Rewriting:

Union of Rule Sets ↑

$\mathcal{R}_1$ has property $\texttt{Prop}$
$\mathcal{R}_2$ has property $\texttt{Prop}$ $\implies$ $\mathcal{R}_1 \cup \mathcal{R}_2$ has property $\texttt{Prop}$

Modularity

Imperative Programs:

Sequential Execution ↑

$\mathcal{P}_1$ has property Prop
$\mathcal{P}_2$ has property Prop $\implies$ $\mathcal{P}_1 ; \mathcal{P}_2$ has property Prop

Term Rewriting:

Union of Rule Sets ↑

$\mathcal{R}_1$ has property Prop
$\mathcal{R}_2$ has property Prop $\implies$ $\mathcal{R}_1 \cup \mathcal{R}_2$ has property Prop

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(Innermost) Termination is not Modular

Termination:

(Innermost) Termination is not Modular

Termination:

$\mathcal{R}_1$: Term

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$$

$\mathcal{R}_2$: Term

$$\mathsf{g} \to \mathsf{a}$$
$$\mathsf{g} \to \mathsf{b}$$

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$\mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{g})$

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$\mathcal{R}_1 \cup \mathcal{R}_2$ not Term

(Innermost) Termination is not Modular

Termination:

$\mathcal{R}_1$: Term
$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$$

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$$\mathsf{g} \to \mathsf{a}$$
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$\mathcal{R}_1 \cup \mathcal{R}_2$ not Term $\Rightarrow$: Termination is not Modular

(Innermost) Termination is not Modular

Termination:

$\mathcal{R}_1$: Term

$$f(a,b,x) \to f(x,x,x)$$

$\mathcal{R}_2$: Term

$$g \to a$$
$$g \to b$$

$$f(a,b,g) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} f(g,g,g) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} f(a,g,g) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} f(a,b,g) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} \dots$$

$\mathcal{R}_1 \cup \mathcal{R}_2$ not Term $\Rightarrow$: Termination is not Modular

Innermost Termination:

(Innermost) Termination is not Modular

Termination:

$\mathcal{R}_1$: Term
$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$$

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$$\mathsf{g} \to \mathsf{a}$$
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$\mathcal{R}_1 \cup \mathcal{R}_2$ not Term $\Rightarrow$: Termination is not Modular

Innermost Termination:

$\mathcal{R}_1$: Term
$$\mathsf{a} \to \mathsf{b}$$

$\mathcal{R}_2$: Term
$$\mathsf{b} \to \mathsf{a}$$

(Innermost) Termination is not Modular

Termination:

$\mathcal{R}_1$: Term

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$$

$\mathcal{R}_2$: Term

$$\mathsf{g} \to \mathsf{a} \qquad \mathsf{g} \to \mathsf{b}$$

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} \mathsf{f}(\mathsf{g},\mathsf{g},\mathsf{g}) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} \mathsf{f}(\mathsf{a},\mathsf{g},\mathsf{g}) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} \ldots$$

$\mathcal{R}_1 \cup \mathcal{R}_2$ not Term $\Rightarrow$: Termination is not Modular

Innermost Termination:

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$$\mathsf{a} \to \mathsf{b}$$

$\mathcal{R}_2$: Term

$$\mathsf{b} \to \mathsf{a}$$

a

(Innermost) Termination is not Modular

Termination:

$\mathcal{R}_1$: Term
$$f(a, b, x) \to f(x, x, x)$$

$\mathcal{R}_2$: Term
$$g \to a$$
$$g \to b$$

$$f(a, b, g) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} f(g, g, g) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} f(a, g, g) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} f(a, b, g) \to_{\mathcal{R}_1 \cup \mathcal{R}_2} \ldots$$

$\mathcal{R}_1 \cup \mathcal{R}_2$ not Term $\Rightarrow$: Termination is not Modular

Innermost Termination:

$\mathcal{R}_1$: Term
$$a \to b$$

$\mathcal{R}_2$: Term
$$b \to a$$

$$a \xrightarrow{i}_{\mathcal{R}_1 \cup \mathcal{R}_2} b$$

(Innermost) Termination is not Modular

Termination:

$\mathcal{R}_1$: Term
$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x)$$

$\mathcal{R}_2$: Term
$$\mathsf{g} \to \mathsf{a} \qquad \mathsf{g} \to \mathsf{b}$$

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$$\mathsf{b} \to \mathsf{a}$$

$$\mathsf{a} \xrightarrow{\mathsf{i}}_{\mathcal{R}_1 \cup \mathcal{R}_2} \mathsf{b} \xrightarrow{\mathsf{i}}_{\mathcal{R}_1 \cup \mathcal{R}_2} \mathsf{a}$$

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$\mathcal{R}_2$: Term
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$$\mathsf{a} \xrightarrow{\mathsf{i}}_{\mathcal{R}_1 \cup \mathcal{R}_2} \mathsf{b} \xrightarrow{\mathsf{i}}_{\mathcal{R}_1 \cup \mathcal{R}_2} \mathsf{a} \xrightarrow{\mathsf{i}}_{\mathcal{R}_1 \cup \mathcal{R}_2} \ldots$$

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(Innermost) Termination is not Modular

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$\mathcal{R}_1 \cup \mathcal{R}_2$ not Term $\Rightarrow$: Innermost Termination is not Modular

Innermost Termination is Modular for ...

Disjoint Unions:

Innermost Termination is Modular for . . .

Disjoint Unions:

$\mathcal{R}'_{len}$: Term

$$\begin{aligned} \mathsf{len}(\mathsf{nil}) &\to 0' \\ \mathsf{len}(\mathsf{cons}(x, y)) &\to \mathsf{s}'(\mathsf{len}(y)) \end{aligned}$$

$\mathcal{R}_{add}$: Term

$$\begin{aligned} \mathsf{plus}(0, x) &\to x \\ \mathsf{plus}(\mathsf{s}(x), y) &\to \mathsf{s}(\mathsf{plus}(x, y)) \end{aligned}$$

Innermost Termination is Modular for ...

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$$\mathsf{len}(\mathsf{cons}(\mathsf{plus}(0,\mathsf{s}(0)),\mathsf{nil}))$$

Innermost Termination is Modular for ...

Disjoint Unions:

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$$\begin{aligned} \mathsf{plus}(\mathsf{0},x) &\to x \\ \mathsf{plus}(\mathsf{s}(x),y) &\to \mathsf{s}(\mathsf{plus}(x,y)) \end{aligned}$$

$$\mathsf{len}(\mathsf{cons}(\mathsf{plus}(\mathsf{0},\mathsf{s}(\mathsf{0})),\mathsf{nil}))$$

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Disjoint Unions:

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$\mathcal{R}_{add}$: Term

$$\begin{aligned} \mathsf{plus}(0, x) &\to x \\ \mathsf{plus}(\mathsf{s}(x), y) &\to \mathsf{s}(\mathsf{plus}(x, y)) \end{aligned}$$

$$\mathsf{len}(\mathsf{cons}(\mathsf{plus}(0, \mathsf{s}(0)), \mathsf{nil})) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{add}} \mathsf{len}(\mathsf{cons}(\mathsf{s}(0), \mathsf{nil}))$$

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Overview

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1. Introduce Probabilistic Notions of Termination:

$$\mathtt{SAST} \subsetneq \mathtt{PAST} \subsetneq \mathtt{AST}$$

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Almost-Sure Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $$\mathsf{g}(0) \to \{\, 1/2 : 0,\ 1/2 : \mathsf{g}(\mathsf{g}(0)) \,\}$$

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$$
\begin{aligned}
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- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \to_{\mathcal{R}} \mu_1 \to_{\mathcal{R}} \ldots$

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Almost-Sure Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\quad \mathsf{g}(0) \to \{\, 1/2 : 0,\ 1/2 : \mathsf{g}(\mathsf{g}(0)) \,\}$

Multi-Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$ $\quad |\mu|$

	$\{\, 1 : \mathsf{g}(0) \,\}$	0
$\to_{\mathcal{R}_{rw}}$	$\{\, 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \,\}$	$1/2$
$\to_{\mathcal{R}_{rw}}$	$\{\, 1/2 : 0,\ 1/4 : \mathsf{g}(0),\ 1/4 : \mathsf{g}^3(0) \,\}$	$1/2$
$\to_{\mathcal{R}_{rw}}$	$\{\, 1/2 : 0,\ 1/8 : 0,\ 1/8 : \mathsf{g}^2(0),\ 1/8 : \mathsf{g}^2(0),\ 1/8 : \mathsf{g}^4(0) \,\}$	$5/8$

Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \to_{\mathcal{R}} \mu_1 \to_{\mathcal{R}} \ldots$ No
- $\mathcal{R}$ is *almost-surely terminating* (AST) iff $\lim_{n\to\infty} |\mu_n| = 1$ for every infinite evaluation $\mu_0 \to_{\mathcal{R}} \mu_1 \to_{\mathcal{R}} \ldots$ Yes

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $$\mathrm{g} \to \{\, 1/2 : 0,\ 1/2 : \mathrm{g} \,\}$$

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\mathsf{g} \to \{\, {}^{1}\!/_{2} : 0,\ {}^{1}\!/_{2} : \mathsf{g} \,\}$

$\{\, 1 : \mathsf{g} \,\}$

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\mathsf{g} \rightarrow \{\, {}^1\!/_2 : \mathsf{0},\ {}^1\!/_2 : \mathsf{g} \,\}$

$$\{\, 1 : \mathsf{g} \,\}$$

$$\rightarrow_{\mathcal{R}_{coin}} \{\, {}^1\!/_2 : \mathsf{0},\ {}^1\!/_2 : \mathsf{g} \,\}$$

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\mathsf{g} \to \{ 1/2 : 0,\ 1/2 : \mathsf{g} \}$

$$
\begin{aligned}
& \{ 1 : \mathsf{g} \} \\
\to_{\mathcal{R}_{coin}}\ & \{ 1/2 : 0,\ 1/2 : \mathsf{g} \} \\
\to_{\mathcal{R}_{coin}}\ & \{ 1/2 : 0,\ 1/4 : 0,\ 1/4 : \mathsf{g} \}
\end{aligned}
$$

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\quad \mathsf{g} \to \{ 1/2 : 0,\ 1/2 : \mathsf{g} \}$

$$
\begin{aligned}
& \{ 1 : \mathsf{g} \} \\
\to_{\mathcal{R}_{coin}} \quad & \{ 1/2 : 0,\ 1/2 : \mathsf{g} \} \\
\to_{\mathcal{R}_{coin}} \quad & \{ 1/2 : 0,\ 1/4 : 0,\ 1/4 : \mathsf{g} \} \\
\to_{\mathcal{R}_{coin}} \quad & \ldots
\end{aligned}
$$

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\quad g \to \{\, ^1/_2 : 0,\ ^1/_2 : g \,\}$

$$
\begin{aligned}
& \{\, 1 : g \,\} \\
\to_{\mathcal{R}_{coin}}\ & \{\, ^1/_2 : 0,\ ^1/_2 : g \,\} \\
\to_{\mathcal{R}_{coin}}\ & \{\, ^1/_2 : 0,\ ^1/_4 : 0,\ ^1/_4 : g \,\} \\
\to_{\mathcal{R}_{coin}}\ & \ldots
\end{aligned}
$$

Positive/Strong Almost-Sure Termination for PTRSs

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\quad \mathsf{g} \to \{\, ^1/_2 : 0,\ ^1/_2 : \mathsf{g} \,\}$

$$\begin{aligned} & \{\, 1 : \mathsf{g} \,\} \\ \to_{\mathcal{R}_{coin}} \quad & \{\, ^1/_2 : 0,\ ^1/_2 : \mathsf{g} \,\} \\ \to_{\mathcal{R}_{coin}} \quad & \{\, ^1/_2 : 0,\ ^1/_4 : 0,\ ^1/_4 : \mathsf{g} \,\} \\ \to_{\mathcal{R}_{coin}} \quad & \ldots \end{aligned}$$

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\mathsf{g} \to \{\, 1/2 : 0,\ 1/2 : \mathsf{g} \,\}$

$$\{\, 1 : \mathsf{g} \,\} \qquad |\mu| = 0$$

$$\to_{\mathcal{R}_{coin}} \{\, 1/2 : 0,\ 1/2 : \mathsf{g} \,\}$$

$$\to_{\mathcal{R}_{coin}} \{\, 1/2 : 0,\ 1/4 : 0,\ 1/4 : \mathsf{g} \,\}$$

$$\to_{\mathcal{R}_{coin}} \ldots$$

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\quad \mathsf{g} \to \{\, ^1\!/_2 : 0,\ ^1\!/_2 : \mathsf{g} \,\}$

$$
\begin{aligned}
& \{\, 1 : \mathsf{g} \,\} && |\mu| = 0 \\
\to_{\mathcal{R}_{coin}} \; & \{\, ^1\!/_2 : 0,\ ^1\!/_2 : \mathsf{g} \,\} && |\mu| = ^1\!/_2 \\
\to_{\mathcal{R}_{coin}} \; & \{\, ^1\!/_2 : 0,\ ^1\!/_4 : 0,\ ^1\!/_4 : \mathsf{g} \,\} \\
\to_{\mathcal{R}_{coin}} \; & \ldots
\end{aligned}
$$

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\mathsf{g} \to \{ 1/2 : 0,\ 1/2 : \mathsf{g} \}$

	$\{ 1 : \mathsf{g} \}$	$\lvert\mu\rvert = 0$
$\to_{\mathcal{R}_{coin}}$	$\{ 1/2 : 0,\ 1/2 : \mathsf{g} \}$	$\lvert\mu\rvert = 1/2$
$\to_{\mathcal{R}_{coin}}$	$\{ 1/2 : 0,\ 1/4 : 0,\ 1/4 : \mathsf{g} \}$	$\lvert\mu\rvert = 3/4$
$\to_{\mathcal{R}_{coin}}$	$\dots$	

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\quad \mathsf{g} \to \{\, {}^{1}/_{2} : 0,\ {}^{1}/_{2} : \mathsf{g} \,\}$

$$\begin{aligned}
& \{\, 1 : \mathsf{g} \,\} && |\mu| = 0 \\
\to_{\mathcal{R}_{coin}} \; & \{\, {}^{1}/_{2} : 0,\ {}^{1}/_{2} : \mathsf{g} \,\} && |\mu| = {}^{1}/_{2} \\
\to_{\mathcal{R}_{coin}} \; & \{\, {}^{1}/_{2} : 0,\ {}^{1}/_{4} : 0,\ {}^{1}/_{4} : \mathsf{g} \,\} && |\mu| = {}^{3}/_{4} \\
\to_{\mathcal{R}_{coin}} \; & \ldots
\end{aligned}$$

$$\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$$

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\mathsf{g} \to \{ 1/2 : 0,\ 1/2 : \mathsf{g} \}$

$$\{ 1 : \mathsf{g} \} \qquad |\mu| = 0$$

$$\to_{\mathcal{R}_{coin}} \{ 1/2 : 0,\ 1/2 : \mathsf{g} \} \qquad |\mu| = 1/2$$

$$\to_{\mathcal{R}_{coin}} \{ 1/2 : 0,\ 1/4 : 0,\ 1/4 : \mathsf{g} \} \qquad |\mu| = 3/4$$

$$\to_{\mathcal{R}_{coin}} \ldots$$

$$\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|) = 1 + 1/2 + 1/4 + \ldots$$

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\quad \mathsf{g} \to \{ 1/2 : 0, \ 1/2 : \mathsf{g} \}$

$$\{ 1 : \mathsf{g} \} \qquad |\mu| = 0$$

$$\to_{\mathcal{R}_{coin}} \{ 1/2 : 0, \ 1/2 : \mathsf{g} \} \qquad |\mu| = 1/2$$

$$\to_{\mathcal{R}_{coin}} \{ 1/2 : 0, \ 1/4 : 0, \ 1/4 : \mathsf{g} \} \qquad |\mu| = 3/4$$

$$\to_{\mathcal{R}_{coin}} \ldots$$

$$\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty} (1 - |\mu_n|) = 1 + 1/2 + 1/4 + \ldots = \sum_{n=0}^{\infty} (1/2)^n$$

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty} (1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\quad \mathsf{g} \to \{ 1/2 : 0,\ 1/2 : \mathsf{g} \}$

$$
\begin{array}{lll}
 & \{ 1 : \mathsf{g} \} & |\mu| = 0 \\
\to_{\mathcal{R}_{coin}} & \{ 1/2 : 0,\ 1/2 : \mathsf{g} \} & |\mu| = 1/2 \\
\to_{\mathcal{R}_{coin}} & \{ 1/2 : 0,\ 1/4 : 0,\ 1/4 : \mathsf{g} \} & |\mu| = 3/4 \\
\to_{\mathcal{R}_{coin}} & \dots &
\end{array}
$$

$$\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|) = 1 + 1/2 + 1/4 + \ldots = \sum_{n=0}^{\infty}(1/2)^n = 2$$

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\mathsf{g} \to \{ 1/2 : 0,\ 1/2 : \mathsf{g} \}$

$$
\begin{aligned}
& \{ 1 : \mathsf{g} \} && |\mu| = 0 \\
\to_{\mathcal{R}_{coin}} & \{ 1/2 : 0,\ 1/2 : \mathsf{g} \} && |\mu| = 1/2 \\
\to_{\mathcal{R}_{coin}} & \{ 1/2 : 0,\ 1/4 : 0,\ 1/4 : \mathsf{g} \} && |\mu| = 3/4 \\
\to_{\mathcal{R}_{coin}} & \dots
\end{aligned}
$$

$$\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|) = 1 + 1/2 + 1/4 + \dots = \sum_{n=0}^{\infty}(1/2)^n = 2$$

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation — Yes

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\mathsf{g} \to \{ {}^1\!/_2 : 0,\ {}^1\!/_2 : \mathsf{g} \}$

$$
\begin{array}{lll}
 & \{ 1 : \mathsf{g} \} & |\mu| = 0 \\
\to_{\mathcal{R}_{coin}} & \{ {}^1\!/_2 : 0,\ {}^1\!/_2 : \mathsf{g} \} & |\mu| = {}^1\!/_2 \\
\to_{\mathcal{R}_{coin}} & \{ {}^1\!/_2 : 0,\ {}^1\!/_4 : 0,\ {}^1\!/_4 : \mathsf{g} \} & |\mu| = {}^3\!/_4 \\
\to_{\mathcal{R}_{coin}} & \ldots &
\end{array}
$$

$$\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|) = 1 + {}^1\!/_2 + {}^1\!/_4 + \ldots = \sum_{n=0}^{\infty}({}^1\!/_2)^n = 2$$

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation Yes
- $\mathcal{R}$ is *Strong almost-surely terminating* (SAST) iff there exists a $C_t \in \mathbb{R}$ such that $\mathbb{E}(\vec{\mu}) < C_t < \infty$ for every infinite evaluation $\vec{\mu}$ starting with $\{1 : t\}$

Positive and Strong Almost-Sure Termination

$\mathcal{R}_{coin}$: $\quad \mathsf{g} \to \{ 1/2 : 0,\ 1/2 : \mathsf{g} \}$

$$\{ 1 : \mathsf{g} \} \qquad |\mu| = 0$$

$$\to_{\mathcal{R}_{coin}} \{ 1/2 : 0,\ 1/2 : \mathsf{g} \} \qquad |\mu| = 1/2$$

$$\to_{\mathcal{R}_{coin}} \{ 1/2 : 0,\ 1/4 : 0,\ 1/4 : \mathsf{g} \} \qquad |\mu| = 3/4$$

$$\to_{\mathcal{R}_{coin}} \ldots$$

$$\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|) = 1 + 1/2 + 1/4 + \ldots = \sum_{n=0}^{\infty}(1/2)^n = 2$$

Positive/Strong Almost-Sure Termination for PTRSs

- $\mathcal{R}$ is *Positive almost-surely terminating* (PAST) iff $\mathbb{E}(\vec{\mu}) = \sum_{n=0}^{\infty}(1 - |\mu_n|)$ (*expected runtime*) is finite for every infinite evaluation Yes
- $\mathcal{R}$ is *Strong almost-surely terminating* (SAST) iff there exists a $C_t \in \mathbb{R}$ such that $\mathbb{E}(\vec{\mu}) < C_t < \infty$ for every infinite evaluation $\vec{\mu}$ starting with $\{1 : t\}$ Yes

AST vs. PAST vs. SAST

$$\mathtt{SAST} \subsetneq \mathtt{PAST} \subsetneq \mathtt{AST}$$

AST vs. PAST vs. SAST

$$\mathtt{SAST} \subsetneq \mathtt{PAST} \subsetneq \mathtt{AST}$$

AST **and not** PAST**:**

AST vs. PAST vs. SAST

$$\mathtt{SAST} \subsetneq \mathtt{PAST} \subsetneq \mathtt{AST}$$

AST and not PAST:

$\mathcal{R}_{rw}$: $$\mathrm{g}(0) \quad \to \quad \{\, 1/2 : 0,\ 1/2 : \mathrm{g}(\mathrm{g}(0))\,\}$$

AST vs. PAST vs. SAST

$$\mathtt{SAST} \subsetneq \mathtt{PAST} \subsetneq \mathtt{AST}$$

AST and not PAST:

$\mathcal{R}_{rw}$: $\quad g(0) \to \{\, 1/2 : 0,\ 1/2 : g(g(0)) \,\}$

$\Rightarrow$ AST as we have seen

AST vs. PAST vs. SAST

$$\mathtt{SAST} \subsetneq \mathtt{PAST} \subsetneq \mathtt{AST}$$

AST and not PAST:

$\mathcal{R}_{rw}$: $\mathsf{g}(0) \to \{\, 1/2 : 0,\ 1/2 : \mathsf{g}(\mathsf{g}(0)) \,\}$

⇒ AST as we have seen

⇒ Not PAST (no details)

AST vs. PAST vs. SAST

PAST **and not** SAST:

AST vs. PAST vs. SAST

PAST and not SAST:

$$
\mathcal{R}: \quad \begin{aligned}
\mathsf{f}(x) &\to \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\
\mathsf{f}(x) &\to \{1 : \mathsf{g}(x)\} \\
\text{“}\mathsf{g}(\mathsf{s}^k(x)) &\to \Theta(4^k)\text{”}
\end{aligned}
$$

AST vs. PAST vs. SAST

PAST and not SAST:

$\mathcal{R}$:

$$
\begin{aligned}
\mathsf{f}(x) &\to \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\
\mathsf{f}(x) &\to \{1 : \mathsf{g}(x)\} \\
\text{``}\mathsf{g}(\mathsf{s}^k(x)) &\to \Theta(4^k)\text{''}
\end{aligned}
$$

Starting with $\{1 : \mathsf{f}(0)\}$:

AST vs. PAST vs. SAST

PAST and not SAST:

$\mathcal{R}$:

$$
\begin{aligned}
\mathsf{f}(x) &\rightarrow \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\
\mathsf{f}(x) &\rightarrow \{1 : \mathsf{g}(x)\} \\
\text{“}\mathsf{g}(\mathsf{s}^k(x)) &\rightarrow \Theta(4^k)\text{”}
\end{aligned}
$$

Starting with $\{1 : \mathsf{f}(0)\}$:

1. Only using the first f-rule:

$$\{1 : \mathsf{f}(0)\} \rightarrow_{\mathcal{R}} \{1/2 : \mathsf{f}(\mathsf{s}(0)), 1/2 : 0\}$$

AST vs. PAST vs. SAST

PAST and not SAST:

$\mathcal{R}$:
$$\begin{array}{rcl} \mathsf{f}(x) & \to & \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) & \to & \{1 : \mathsf{g}(x)\} \\ \text{“}\mathsf{g}(\mathsf{s}^k(x)) & \to & \Theta(4^k)\text{”} \end{array}$$

Starting with $\{1 : \mathsf{f}(0)\}$:

1. Only using the first f-rule:

$$\{1 : \mathsf{f}(0)\} \to_{\mathcal{R}} \{1/2 : \mathsf{f}(\mathsf{s}(0)), 1/2 : 0\}$$
$$\to_{\mathcal{R}} \{1/4 : \mathsf{f}(\mathsf{s}^2(0)), 3/4 : 0\} \to_{\mathcal{R}} \dots$$

AST vs. PAST vs. SAST

PAST and not SAST:

$$\mathcal{R}: \quad \begin{array}{rcl} \mathsf{f}(x) & \to & \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) & \to & \{1 : \mathsf{g}(x)\} \\ \text{``}\mathsf{g}(\mathsf{s}^k(x)) & \to & \Theta(4^k)\text{''} \end{array}$$

Starting with $\{1 : \mathsf{f}(0)\}$:

1. Only using the first f-rule:

$$\{1 : \mathsf{f}(0)\} \to_{\mathcal{R}} \{1/2 : \mathsf{f}(\mathsf{s}(0)), 1/2 : 0\}$$
$$\to_{\mathcal{R}} \{1/4 : \mathsf{f}(\mathsf{s}^2(0)), 3/4 : 0\} \to_{\mathcal{R}} \ldots$$

Coin Flip $\Rightarrow \mathbb{E}(\vec{\mu}) = 2 < \infty$

AST vs. PAST vs. SAST

PAST and not SAST:

$\mathcal{R}$:

$$\begin{aligned} \mathsf{f}(x) &\to \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{g}(x)\} \\ \text{``}\mathsf{g}(\mathsf{s}^k(x)) &\to \Theta(4^k)\text{''} \end{aligned}$$

Starting with $\{1 : \mathsf{f}(0)\}$:

1. Only using the first f-rule:

$$\begin{aligned} \{1 : \mathsf{f}(0)\} &\to_{\mathcal{R}} \{1/2 : \mathsf{f}(\mathsf{s}(0)), 1/2 : 0\} \\ &\to_{\mathcal{R}} \{1/4 : \mathsf{f}(\mathsf{s}^2(0)), 3/4 : 0\} \to_{\mathcal{R}} \dots \end{aligned}$$

   Coin Flip $\Rightarrow \mathbb{E}(\vec{\mu}) = 2 < \infty$

2. Using the first f-rule k-times:

$$\{1 : \mathsf{f}(0)\}$$

AST vs. PAST vs. SAST

PAST and not SAST:

$$\mathcal{R}: \quad \begin{array}{rcl} \mathsf{f}(x) & \to & \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) & \to & \{1 : \mathsf{g}(x)\} \\ \text{“}\mathsf{g}(\mathsf{s}^k(x)) & \to & \Theta(4^k)\text{”} \end{array}$$

Starting with $\{1 : \mathsf{f}(0)\}$:

1. Only using the first f-rule:

$$\{1 : \mathsf{f}(0)\} \to_{\mathcal{R}} \{1/2 : \mathsf{f}(\mathsf{s}(0)), 1/2 : 0\}$$
$$\to_{\mathcal{R}} \{1/4 : \mathsf{f}(\mathsf{s}^2(0)), 3/4 : 0\} \to_{\mathcal{R}} \dots$$

   Coin Flip $\Rightarrow \mathbb{E}(\vec{\mu}) = 2 < \infty$

2. Using the first f-rule k-times:

$$\{1 : \mathsf{f}(0)\} \to_{\mathcal{R}}^{k} \{(1/2)^k : \mathsf{f}(\mathsf{s}^k(0)), 1 - (1/2)^k : 0\}$$

AST vs. PAST vs. SAST

PAST and not SAST:

$$\mathcal{R}: \quad \begin{aligned} f(x) &\to \{1/2 : f(s(x)), 1/2 : 0\} \\ f(x) &\to \{1 : g(x)\} \\ \text{“}g(s^k(x)) &\to \Theta(4^k)\text{”} \end{aligned}$$

Starting with $\{1 : f(0)\}$:

1. Only using the first f-rule:

$$\{1 : f(0)\} \to_{\mathcal{R}} \{1/2 : f(s(0)), 1/2 : 0\}$$
$$\to_{\mathcal{R}} \{1/4 : f(s^2(0)), 3/4 : 0\} \to_{\mathcal{R}} \dots$$

   Coin Flip $\Rightarrow \mathbb{E}(\vec{\mu}) = 2 < \infty$

2. Using the first f-rule k-times:

$$\{1 : f(0)\} \to_{\mathcal{R}}^{k} \{(1/2)^k : f(s^k(0)), 1 - (1/2)^k : 0\}$$
$$\to_{\mathcal{R}} \{(1/2)^k : g(s^k(0)), 1 - (1/2)^k : 0\} \to_{\mathcal{R}} \dots$$

AST vs. PAST vs. SAST

PAST and not SAST:

$$\mathcal{R}: \quad \begin{array}{rcl} \mathsf{f}(x) & \to & \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) & \to & \{1 : \mathsf{g}(x)\} \\ \text{``}\mathsf{g}(\mathsf{s}^k(x)) & \to & \Theta(4^k)\text{''} \end{array}$$

Starting with $\{1 : \mathsf{f}(0)\}$:

1. Only using the first f-rule:

$$\{1 : \mathsf{f}(0)\} \to_{\mathcal{R}} \{1/2 : \mathsf{f}(\mathsf{s}(0)), 1/2 : 0\}$$
$$\to_{\mathcal{R}} \{1/4 : \mathsf{f}(\mathsf{s}^2(0)), 3/4 : 0\} \to_{\mathcal{R}} \dots$$

   Coin Flip $\Rightarrow \mathbb{E}(\vec{\mu}) = 2 < \infty$

2. Using the first f-rule k-times:

$$\{1 : \mathsf{f}(0)\} \to_{\mathcal{R}}^{k} \{(1/2)^k : \mathsf{f}(\mathsf{s}^k(0)), 1 - (1/2)^k : 0\}$$
$$\to_{\mathcal{R}} \{(1/2)^k : \mathsf{g}(\mathsf{s}^k(0)), 1 - (1/2)^k : 0\} \to_{\mathcal{R}} \dots$$

AST vs. PAST vs. SAST

PAST and not SAST:

$\mathcal{R}$:
$$\begin{aligned} f(x) &\to \{1/2 : f(s(x)), 1/2 : 0\} \\ f(x) &\to \{1 : g(x)\} \\ \text{“}g(s^k(x)) &\to \Theta(4^k)\text{”} \end{aligned}$$

Starting with $\{1 : f(0)\}$:

1. Only using the first f-rule:

$$\begin{aligned} \{1 : f(0)\} &\to_{\mathcal{R}} \{1/2 : f(s(0)), 1/2 : 0\} \\ &\to_{\mathcal{R}} \{1/4 : f(s^2(0)), 3/4 : 0\} \to_{\mathcal{R}} \dots \end{aligned}$$

Coin Flip $\Rightarrow \mathbb{E}(\vec{\mu}) = 2 < \infty$

2. Using the first f-rule k-times:

$$\begin{aligned} \{1 : f(0)\} &\to_{\mathcal{R}}^k \{(1/2)^k : f(s^k(0)), 1 - (1/2)^k : 0\} \\ &\to_{\mathcal{R}} \{(1/2)^k : g(s^k(0)), 1 - (1/2)^k : 0\} \to_{\mathcal{R}} \dots \end{aligned}$$

$\mathbb{E}(\vec{\mu})$

AST vs. PAST vs. SAST

PAST and not SAST:

$$\mathcal{R}: \quad \begin{array}{rcl} \mathsf{f}(x) & \to & \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) & \to & \{1 : \mathsf{g}(x)\} \\ \text{“}\mathsf{g}(\mathsf{s}^k(x)) & \to & \Theta(4^k)\text{”} \end{array}$$

Starting with $\{1 : \mathsf{f}(0)\}$:

1. Only using the first f-rule:

$$\{1 : \mathsf{f}(0)\} \to_{\mathcal{R}} \{1/2 : \mathsf{f}(\mathsf{s}(0)), 1/2 : 0\} \to_{\mathcal{R}} \{1/4 : \mathsf{f}(\mathsf{s}^2(0)), 3/4 : 0\} \to_{\mathcal{R}} \dots$$

   Coin Flip $\Rightarrow \mathbb{E}(\vec{\mu}) = 2 < \infty$

2. Using the first f-rule k-times:

$$\{1 : \mathsf{f}(0)\} \to_{\mathcal{R}}^{k} \{(1/2)^k : \mathsf{f}(\mathsf{s}^k(0)), 1 - (1/2)^k : 0\} \to_{\mathcal{R}} \{(1/2)^k : \mathsf{g}(\mathsf{s}^k(0)), 1 - (1/2)^k : 0\} \to_{\mathcal{R}} \dots$$

   $\mathbb{E}(\vec{\mu}) \approx (1/2)^k \cdot 4^k = 2^k < \infty$

AST vs. PAST vs. SAST

PAST and not SAST:

$\mathcal{R}$:

$$\begin{array}{rcl} f(x) & \to & \{1/2 : f(s(x)), 1/2 : 0\} \\ f(x) & \to & \{1 : g(x)\} \\ \text{“}g(s^k(x)) & \to & \Theta(4^k)\text{”} \end{array}$$

Starting with $\{1 : f(0)\}$:

1. Only using the first f-rule:

$$\{1 : f(0)\} \to_{\mathcal{R}} \{1/2 : f(s(0)), 1/2 : 0\} \to_{\mathcal{R}} \{1/4 : f(s^2(0)), 3/4 : 0\} \to_{\mathcal{R}} \dots$$

   Coin Flip $\Rightarrow \mathbb{E}(\vec{\mu}) = 2 < \infty$

2. Using the first f-rule k-times:

$$\{1 : f(0)\} \to_{\mathcal{R}}^{k} \{(1/2)^k : f(s^k(0)), 1 - (1/2)^k : 0\} \to_{\mathcal{R}} \{(1/2)^k : g(s^k(0)), 1 - (1/2)^k : 0\} \to_{\mathcal{R}} \dots$$

   $\mathbb{E}(\vec{\mu}) \approx (1/2)^k \cdot 4^k = 2^k < \infty$ but unbounded!

Overview

1. Introduce Probabilistic Notions of Termination:

$$\mathtt{SAST} \subsetneq \mathtt{PAST} \subsetneq \mathtt{AST}$$

2. Modularity of $\mathtt{AST}$, $\mathtt{PAST}$, and $\mathtt{SAST}$

3. $\mathtt{PAST} \approx \mathtt{SAST}$ for PTRSs

Overview

1. Introduce Probabilistic Notions of Termination:

$$\mathtt{SAST} \subsetneq \mathtt{PAST} \subsetneq \mathtt{AST}$$

2. Modularity of AST, PAST, and SAST

3. PAST $\approx$ SAST for PTRSs

Modularity AST

Modularity AST

Disjoint Unions: Yes

$\mathcal{R}_1$: AST

$$f(x) \to \{1/2 : x, 1/2 : f^2(x)\}$$

$\mathcal{R}_2$: AST

$$g(x) \to \{1/2 : x, 1/2 : g^2(x)\}$$

Modularity AST

Disjoint Unions: Yes

$\mathcal{R}_1$: AST

$$f(x) \to \{1/2 : x, 1/2 : f^2(x)\}$$

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$$g(x) \to \{1/2 : x, 1/2 : g^2(x)\}$$

$$f(g(x))$$

Modularity AST

Disjoint Unions: Yes

$\mathcal{R}_1$: AST
$$f(x) \to \{1/2 : x, 1/2 : f^2(x)\}$$

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$$g(x) \to \{1/2 : x, 1/2 : g^2(x)\}$$

$$f(g(x))$$

Modularity AST

Disjoint Unions: Yes

$\mathcal{R}_1$: AST
$\mathrm{f}(x) \to \{1/2 : x, 1/2 : \mathrm{f}^2(x)\}$

$\mathcal{R}_2$: AST
$\mathrm{g}(x) \to \{1/2 : x, 1/2 : \mathrm{g}^2(x)\}$

$$\mathrm{f}(\mathrm{g}(x))$$

Shared Constructor Systems: Yes

Modularity AST

Disjoint Unions: Yes

$\mathcal{R}_1$: AST
$f(x) \to \{1/2 : x, 1/2 : f^2(x)\}$

$\mathcal{R}_2$: AST
$g(x) \to \{1/2 : x, 1/2 : g^2(x)\}$

$$f(g(x))$$

Shared Constructor Systems: Yes

$\mathcal{R}_1$: AST
$f(s(x)) \to \{1/2 : f(x), 1/2 : f(s^2(x))\}$

$\mathcal{R}_2$: AST
$g(0) \to \{1/2 : s(0), 1/2 : s(g^2(0))\}$

Modularity AST

Disjoint Unions: Yes

$\mathcal{R}_1$: AST
$f(x) \to \{1/2 : x, 1/2 : f^2(x)\}$

$\mathcal{R}_2$: AST
$g(x) \to \{1/2 : x, 1/2 : g^2(x)\}$

$$f(g(x))$$

Shared Constructor Systems: Yes

$\mathcal{R}_1$: AST
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$\mathcal{R}_2$: AST
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$$\{1 : f(g(0))\}$$

Modularity AST

Disjoint Unions: Yes

$\mathcal{R}_1$: AST
$f(x) \to \{1/2 : x, 1/2 : f^2(x)\}$

$\mathcal{R}_2$: AST
$g(x) \to \{1/2 : x, 1/2 : g^2(x)\}$

$$f(g(x))$$

Shared Constructor Systems: Yes

$\mathcal{R}_1$: AST
$f(s(x)) \to \{1/2 : f(x), 1/2 : f(s^2(x))\}$

$\mathcal{R}_2$: AST
$g(0) \to \{1/2 : s(0), 1/2 : s(g^2(0))\}$

$$\{1 : f(g(0))\} \xrightarrow{i}_{\mathcal{R}_2} \{1/2 : f(s(0)), 1/2 : f(s(g(g(0))))\}$$

Modularity AST

Disjoint Unions: Yes

$\mathcal{R}_1$: AST
$f(x) \to \{1/2 : x, 1/2 : f^2(x)\}$

$\mathcal{R}_2$: AST
$g(x) \to \{1/2 : x, 1/2 : g^2(x)\}$

$$f(g(x))$$

Shared Constructor Systems: Yes

$\mathcal{R}_1$: AST
$f(s(x)) \to \{1/2 : f(x), 1/2 : f(s^2(x))\}$

$\mathcal{R}_2$: AST
$g(0) \to \{1/2 : s(0), 1/2 : s(g^2(0))\}$

$$\{1 : f(g(0))\} \xrightarrow{i}_{\mathcal{R}_2} \{1/2 : f(s(0)), 1/2 : f(s(g(g(0))))\} \xrightarrow{i}_{\mathcal{R}_1} \dots$$

Modularity PAST

Modularity PAST

Disjoint Unions: No

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{aligned} \mathsf{f}(x) &\to \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{g}(x)\} \\ \text{``}\mathsf{g}(\mathsf{s}^k(x)) &\to \Theta(4^k)\text{''} \end{aligned}$$

$\mathcal{R}_2$: PAST

$$\mathsf{b}(x) \to \mathsf{c}(x, x)$$

$$\{1 : \mathsf{c}(\mathsf{f}(0), \mathsf{f}(0))\}$$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{array}{rcl} \mathsf{f}(x) & \to & \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) & \to & \{1 : \mathsf{g}(x)\} \\ \text{“}\mathsf{g}(\mathsf{s}^k(x)) & \to & \Theta(4^k)\text{”} \end{array}$$

$\mathcal{R}_2$: PAST

$$\mathsf{b}(x) \to \mathsf{c}(x, x)$$

$$\begin{array}{rl} & \{1 : \mathsf{c}(\mathsf{f}(0), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{1/2 : \mathsf{c}(0, \mathsf{f}(0)), 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{s}(0)), \mathsf{f}(0))\} \end{array}$$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$
\begin{array}{rcl}
f(x) & \to & \{1/2 : f(s(x)), 1/2 : 0\} \\
f(x) & \to & \{1 : g(x)\} \\
\text{“}g(s^k(x)) & \to & \Theta(4^k)\text{”}
\end{array}
$$

$\mathcal{R}_2$: PAST

$$
b(x) \to c(x, x)
$$

$$
\begin{array}{ll}
 & \{1 : c(f(0), f(0))\} \\
\to_{\mathcal{R}_1} & \{1/2 : c(0, f(0)), 1/2 : c(f(s(0)), f(0))\} \\
\to_{\mathcal{R}_1} & \{ \quad \dots \quad , 1/4 : c(0, f(0)), 1/4 : c(f(s^2(0)), f(0))\}
\end{array}
$$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{aligned} f(x) &\to \{1/2 : f(s(x)), 1/2 : 0\} \\ f(x) &\to \{1 : g(x)\} \\ \text{“}g(s^k(x)) &\to \Theta(4^k)\text{”} \end{aligned}$$

$\mathcal{R}_2$: PAST

$$b(x) \to c(x, x)$$

$$\begin{aligned} & \{1 : c(f(0), f(0))\} \\ \to_{\mathcal{R}_1} & \{1/2 : c(0, f(0)), 1/2 : c(f(s(0)), f(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \dots \quad , 1/4 : c(0, f(0)), 1/4 : c(f(s^2(0)), f(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \dots \quad , \quad \dots \quad , 1/8 : c(0, f(0)), 1/8 : c(f(s^3(0)), f(0))\} \end{aligned}$$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{aligned} f(x) &\to \{1/2 : f(s(x)), 1/2 : 0\} \\ f(x) &\to \{1 : g(x)\} \\ \text{“}g(s^k(x)) &\to \Theta(4^k)\text{”} \end{aligned}$$

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$$b(x) \to c(x, x)$$

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$\mathbb{E}(\vec{\mu})$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{aligned} f(x) &\to \{1/2 : f(s(x)), 1/2 : 0\} \\ f(x) &\to \{1 : g(x)\} \\ \text{“}g(s^k(x)) &\to \Theta(4^k)\text{”} \end{aligned}$$

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$$b(x) \to c(x, x)$$

$$\begin{aligned} & \{1 : c(f(0), f(0))\} \\ \to_{\mathcal{R}_1} & \{1/2 : c(0, f(0)), 1/2 : c(f(s(0)), f(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \dots \quad, 1/4 : c(0, f(0)), 1/4 : c(f(s^2(0)), f(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \dots \quad, \quad \dots \quad, 1/8 : c(0, f(0)), 1/8 : c(f(s^3(0)), f(0))\} \end{aligned}$$

$$\mathbb{E}(\vec{\mu}) \geq 1/2 \cdot 2^1$$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{aligned} f(x) &\to \{1/2 : f(s(x)), 1/2 : 0\} \\ f(x) &\to \{1 : g(x)\} \\ \text{“}g(s^k(x)) &\to \Theta(4^k)\text{”} \end{aligned}$$

$\mathcal{R}_2$: PAST

$$b(x) \to c(x, x)$$

$$\begin{aligned} & \{1 : c(f(0), f(0))\} \\ \to_{\mathcal{R}_1} \; & \{1/2 : c(0, f(0)), 1/2 : c(f(s(0)), f(0))\} \\ \to_{\mathcal{R}_1} \; & \{ \quad \dots \quad , 1/4 : c(0, f(0)), 1/4 : c(f(s^2(0)), f(0))\} \\ \to_{\mathcal{R}_1} \; & \{ \quad \dots \quad , \quad \dots \quad , 1/8 : c(0, f(0)), 1/8 : c(f(s^3(0)), f(0))\} \end{aligned}$$

$$\mathbb{E}(\vec{\mu}) \geq 1/2 \cdot 2^1 + 1/4 \cdot 2^2 +$$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{array}{rcl} \mathsf{f}(x) & \to & \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) & \to & \{1 : \mathsf{g}(x)\} \\ \text{``}\mathsf{g}(\mathsf{s}^k(x)) & \to & \Theta(4^k)\text{''} \end{array}$$

$\mathcal{R}_2$: PAST

$$\mathsf{b}(x) \to \mathsf{c}(x, x)$$

$$\begin{array}{rl} & \{1 : \mathsf{c}(\mathsf{f}(0), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{1/2 : \mathsf{c}(0, \mathsf{f}(0)), 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{s}(0)), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \ldots \quad , 1/4 : \mathsf{c}(0, \mathsf{f}(0)), 1/4 : \mathsf{c}(\mathsf{f}(\mathsf{s}^2(0)), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \ldots \quad , \quad \ldots \quad , 1/8 : \mathsf{c}(0, \mathsf{f}(0)), 1/8 : \mathsf{c}(\mathsf{f}(\mathsf{s}^3(0)), \mathsf{f}(0))\} \end{array}$$

$$\mathbb{E}(\vec{\mu}) \geq 1/2 \cdot 2^1 + 1/4 \cdot 2^2 + 1/8 \cdot 2^3$$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{aligned} \mathsf{f}(x) &\to \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{g}(x)\} \\ \text{“}\mathsf{g}(\mathsf{s}^k(x)) &\to \Theta(4^k)\text{”} \end{aligned}$$

$\mathcal{R}_2$: PAST

$$\mathsf{b}(x) \to \mathsf{c}(x, x)$$

$$\begin{aligned} & \{1 : \mathsf{c}(\mathsf{f}(0), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{1/2 : \mathsf{c}(0, \mathsf{f}(0)), 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{s}(0)), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \ldots \quad , 1/4 : \mathsf{c}(0, \mathsf{f}(0)), 1/4 : \mathsf{c}(\mathsf{f}(\mathsf{s}^2(0)), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \ldots \quad , \quad \ldots \quad , 1/8 : \mathsf{c}(0, \mathsf{f}(0)), 1/8 : \mathsf{c}(\mathsf{f}(\mathsf{s}^3(0)), \mathsf{f}(0))\} \end{aligned}$$

$$\mathbb{E}(\vec{\mu}) \geq 1/2 \cdot 2^1 + 1/4 \cdot 2^2 + 1/8 \cdot 2^3 = \sum_{k=0}^{\infty} (1/2)^k \cdot 2^k$$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{aligned} \mathsf{f}(x) &\to \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{g}(x)\} \\ \text{“}\mathsf{g}(\mathsf{s}^k(x)) &\to \Theta(4^k)\text{”} \end{aligned}$$

$\mathcal{R}_2$: PAST

$$\mathsf{b}(x) \to \mathsf{c}(x, x)$$

$$\begin{aligned} & \{1 : \mathsf{c}(\mathsf{f}(0), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1}\ & \{1/2 : \mathsf{c}(0, \mathsf{f}(0)), 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{s}(0)), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1}\ & \{\ \ldots\ , 1/4 : \mathsf{c}(0, \mathsf{f}(0)), 1/4 : \mathsf{c}(\mathsf{f}(\mathsf{s}^2(0)), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1}\ & \{\ \ldots\ ,\ \ldots\ , 1/8 : \mathsf{c}(0, \mathsf{f}(0)), 1/8 : \mathsf{c}(\mathsf{f}(\mathsf{s}^3(0)), \mathsf{f}(0))\} \end{aligned}$$

$$\mathbb{E}(\vec{\mu}) \geq 1/2 \cdot 2^1 + 1/4 \cdot 2^2 + 1/8 \cdot 2^3 = \sum_{k=0}^{\infty} (1/2)^k \cdot 2^k = \sum_{k=0}^{\infty} 1$$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{aligned} \mathsf{f}(x) &\to \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{g}(x)\} \\ \text{``}\mathsf{g}(\mathsf{s}^k(x)) &\to \Theta(4^k)\text{''} \end{aligned}$$

$\mathcal{R}_2$: PAST

$$\mathsf{b}(x) \to \mathsf{c}(x, x)$$

$$\begin{aligned} & \{1 : \mathsf{c}(\mathsf{f}(0), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{1/2 : \mathsf{c}(0, \mathsf{f}(0)), 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{s}(0)), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \dots \quad , 1/4 : \mathsf{c}(0, \mathsf{f}(0)), 1/4 : \mathsf{c}(\mathsf{f}(\mathsf{s}^2(0)), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \dots \quad , \quad \dots \quad , 1/8 : \mathsf{c}(0, \mathsf{f}(0)), 1/8 : \mathsf{c}(\mathsf{f}(\mathsf{s}^3(0)), \mathsf{f}(0))\} \end{aligned}$$

$$\mathbb{E}(\vec{\mu}) \geq 1/2 \cdot 2^1 + 1/4 \cdot 2^2 + 1/8 \cdot 2^3 = \sum_{k=0}^{\infty} (1/2)^k \cdot 2^k = \sum_{k=0}^{\infty} 1 = \infty$$

Modularity PAST

Disjoint Unions: No

$\mathcal{R}_1$: PAST

$$\begin{array}{rcl} \mathsf{f}(x) & \to & \{1/2 : \mathsf{f}(\mathsf{s}(x)), 1/2 : 0\} \\ \mathsf{f}(x) & \to & \{1 : \mathsf{g}(x)\} \\ \text{``}\mathsf{g}(\mathsf{s}^k(x)) & \to & \Theta(4^k)\text{''} \end{array}$$

$\mathcal{R}_2$: PAST

$$\mathsf{b}(x) \to \mathsf{c}(x, x)$$

$$\begin{array}{ll} & \{1 : \mathsf{c}(\mathsf{f}(0), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{1/2 : \mathsf{c}(0, \mathsf{f}(0)), 1/2 : \mathsf{c}(\mathsf{f}(\mathsf{s}(0)), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \ldots \quad , 1/4 : \mathsf{c}(0, \mathsf{f}(0)), 1/4 : \mathsf{c}(\mathsf{f}(\mathsf{s}^2(0)), \mathsf{f}(0))\} \\ \to_{\mathcal{R}_1} & \{ \quad \ldots \quad , \quad \ldots \quad , 1/8 : \mathsf{c}(0, \mathsf{f}(0)), 1/8 : \mathsf{c}(\mathsf{f}(\mathsf{s}^3(0)), \mathsf{f}(0))\} \end{array}$$

$$\mathbb{E}(\vec{\mu}) \geq 1/2 \cdot 2^1 + 1/4 \cdot 2^2 + 1/8 \cdot 2^3 = \sum_{k=0}^{\infty} (1/2)^k \cdot 2^k = \sum_{k=0}^{\infty} 1 = \infty$$

Shared Constructor Systems: No

Modularity SAST

Modularity SAST

Disjoint Unions: Yes (no details)

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

$\mathcal{R}_1$: SAST

$$\begin{aligned} \mathsf{f}(\mathsf{c}(x,y)) &\to \{1 : \mathsf{c}(\mathsf{f}(x),\mathsf{f}(y))\} \\ \mathsf{f}(0) &\to \{1 : 0\} \end{aligned}$$

$\mathcal{R}_2$: SAST

$$\begin{aligned} \mathsf{g}(x) &\to \{1/2 : \mathsf{g}(\mathsf{d}(x)), 3/4 : x\} \\ \mathsf{d}(x) &\to \{1 : \mathsf{c}(x,x)\} \end{aligned}$$

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

$\mathcal{R}_1$: SAST

$$\begin{aligned} \mathsf{f}(\mathsf{c}(x,y)) &\to \{1 : \mathsf{c}(\mathsf{f}(x),\mathsf{f}(y))\} \\ \mathsf{f}(0) &\to \{1 : 0\} \end{aligned}$$

$\mathcal{R}_2$: SAST

$$\begin{aligned} \mathsf{g}(x) &\to \{{}^1/_2 : \mathsf{g}(\mathsf{d}(x)), {}^3/_4 : x\} \\ \mathsf{d}(x) &\to \{1 : \mathsf{c}(x,x)\} \end{aligned}$$

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

$\mathcal{R}_1$: SAST

$$\mathsf{f}(\mathsf{c}(x,y)) \to \{1 : \mathsf{c}(\mathsf{f}(x),\mathsf{f}(y))\}$$
$$\mathsf{f}(0) \to \{1 : 0\}$$

$\mathcal{R}_2$: SAST

$$\mathsf{g}(x) \to \{1/2 : \mathsf{g}(\mathsf{d}(x)), 3/4 : x\}$$
$$\mathsf{d}(x) \to \{1 : \mathsf{c}(x,x)\}$$

$$\{1 : \mathsf{f}(\mathsf{g}(0))\}$$

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

$\mathcal{R}_1$: SAST

$$f(c(x,y)) \to \{1 : c(f(x), f(y))\}$$
$$f(0) \to \{1 : 0\}$$

$\mathcal{R}_2$: SAST

$$g(x) \to \{1/2 : g(d(x)), 3/4 : x\}$$
$$d(x) \to \{1 : c(x,x)\}$$

$$\begin{aligned} & \{1 : f(g(0))\} \\ \to_{\mathcal{R}_2}^{k} \quad & \{\dots, (1/2)^k : f(d^{k-1}(0)), \dots\} \end{aligned}$$

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

$\mathcal{R}_1$: SAST

$$\mathsf{f}(\mathsf{c}(x,y)) \to \{1 : \mathsf{c}(\mathsf{f}(x),\mathsf{f}(y))\}$$
$$\mathsf{f}(0) \to \{1 : 0\}$$

$\mathcal{R}_2$: SAST

$$\mathsf{g}(x) \to \{1/2 : \mathsf{g}(\mathsf{d}(x)), 3/4 : x\}$$
$$\mathsf{d}(x) \to \{1 : \mathsf{c}(x,x)\}$$

$$\begin{array}{ll} & \{1 : \mathsf{f}(\mathsf{g}(0))\} \\ \to_{\mathcal{R}_2}^{k} & \{\dots, (1/2)^k : \mathsf{f}(\mathsf{d}^{k-1}(0)), \dots\} \\ \to_{\mathcal{R}_2}^{k} & \{\dots, (1/2)^k : \mathsf{f}(\mathsf{c}^{k-1}(0)), \dots\} \end{array}$$

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

$\mathcal{R}_1$: SAST

$$\begin{array}{rcl} \mathsf{f}(\mathsf{c}(x,y)) & \to & \{1 : \mathsf{c}(\mathsf{f}(x),\mathsf{f}(y))\} \\ \mathsf{f}(0) & \to & \{1 : 0\} \end{array}$$

$\mathcal{R}_2$: SAST

$$\begin{array}{rcl} \mathsf{g}(x) & \to & \{1/2 : \mathsf{g}(\mathsf{d}(x)), 3/4 : x\} \\ \mathsf{d}(x) & \to & \{1 : \mathsf{c}(x,x)\} \end{array}$$

$$\begin{array}{ll} & \{1 : \mathsf{f}(\mathsf{g}(0))\} \\ \to^{k}_{\mathcal{R}_2} & \{\ldots, (1/2)^k : \mathsf{f}(\mathsf{d}^{k-1}(0)), \ldots\} \\ \to^{k}_{\mathcal{R}_2} & \{\ldots, (1/2)^k : \mathsf{f}(\mathsf{c}^{k-1}(0)), \ldots\} \\ \to^{2^{k-1}-1}_{\mathcal{R}_1} & \{\ldots, (1/2)^k : \mathsf{c}^{k-1}(0) \quad , \ldots\} \end{array}$$

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

$\mathcal{R}_1$: SAST

$$\begin{array}{rcl} f(c(x,y)) & \to & \{1 : c(f(x), f(y))\} \\ f(0) & \to & \{1 : 0\} \end{array}$$

$\mathcal{R}_2$: SAST

$$\begin{array}{rcl} g(x) & \to & \{1/2 : g(d(x)), 3/4 : x\} \\ d(x) & \to & \{1 : c(x,x)\} \end{array}$$

$$\begin{array}{ll} & \{1 : f(g(0))\} \\ \to^{k}_{\mathcal{R}_2} & \{\dots, (1/2)^k : f(d^{k-1}(0)), \dots\} \\ \to^{k}_{\mathcal{R}_2} & \{\dots, (1/2)^k : f(c^{k-1}(0)), \dots\} \\ \to^{2^{k-1}-1}_{\mathcal{R}_1} & \{\dots, (1/2)^k : c^{k-1}(0) \quad , \dots\} \end{array}$$

$\mathbb{E}(\vec{\mu})$

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

$\mathcal{R}_1$: SAST

$$\begin{aligned} \mathsf{f}(\mathsf{c}(x,y)) &\to \{1 : \mathsf{c}(\mathsf{f}(x), \mathsf{f}(y))\} \\ \mathsf{f}(0) &\to \{1 : 0\} \end{aligned}$$

$\mathcal{R}_2$: SAST

$$\begin{aligned} \mathsf{g}(x) &\to \{1/2 : \mathsf{g}(\mathsf{d}(x)), 3/4 : x\} \\ \mathsf{d}(x) &\to \{1 : \mathsf{c}(x,x)\} \end{aligned}$$

$$\begin{aligned} & \{1 : \mathsf{f}(\mathsf{g}(0))\} \\ \to^{k}_{\mathcal{R}_2} \quad & \{\dots, (1/2)^k : \mathsf{f}(\mathsf{d}^{k-1}(0)), \dots\} \\ \to^{k}_{\mathcal{R}_2} \quad & \{\dots, (1/2)^k : \mathsf{f}(\mathsf{c}^{k-1}(0)), \dots\} \\ \to^{2^{k-1}-1}_{\mathcal{R}_1} \quad & \{\dots, (1/2)^k : \mathsf{c}^{k-1}(0) \quad , \dots\} \end{aligned}$$

$$\mathbb{E}(\vec{\mu}) \geq \frac{1}{2} \cdot 1 + (1/2)^2 \cdot 2 + (1/2)^3 \cdot 2^2$$

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

$\mathcal{R}_1$: SAST

$$\begin{aligned} \mathsf{f}(\mathsf{c}(x,y)) &\to \{1 : \mathsf{c}(\mathsf{f}(x),\mathsf{f}(y))\} \\ \mathsf{f}(0) &\to \{1 : 0\} \end{aligned}$$

$\mathcal{R}_2$: SAST

$$\begin{aligned} \mathsf{g}(x) &\to \{1/2 : \mathsf{g}(\mathsf{d}(x)), 3/4 : x\} \\ \mathsf{d}(x) &\to \{1 : \mathsf{c}(x,x)\} \end{aligned}$$

$$\begin{aligned} & \{1 : \mathsf{f}(\mathsf{g}(0))\} \\ \to^{k}_{\mathcal{R}_2} \quad & \{\dots, (1/2)^k : \mathsf{f}(\mathsf{d}^{k-1}(0)), \dots\} \\ \to^{k}_{\mathcal{R}_2} \quad & \{\dots, (1/2)^k : \mathsf{f}(\mathsf{c}^{k-1}(0)), \dots\} \\ \to^{2^{k-1}-1}_{\mathcal{R}_1} \quad & \{\dots, (1/2)^k : \mathsf{c}^{k-1}(0) \quad , \dots\} \end{aligned}$$

$$\mathbb{E}(\vec{\mu}) \geq \frac{1}{2} \cdot 1 + (1/2)^2 \cdot 2 + (1/2)^3 \cdot 2^2 = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \cdot 2^n$$

Modularity SAST

Disjoint Unions: Yes (no details)

Shared Constructor Systems: No

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Modularity for PTRSs

Innermost Rewriting with ...	AST	PAST	SAST
Disjoint Unions	Yes	No	Yes
Shared Constructors	Yes	No	No

Overview

1. Introduce Probabilistic Notions of Termination:

$$\mathtt{SAST} \subsetneq \mathtt{PAST} \subsetneq \mathtt{AST}$$

2. Modularity of AST, PAST, and SAST

3. PAST $\approx$ SAST for PTRSs

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$$\mathbb{E}(\vec{\mu}) \geq 1/2 \cdot 2^1 + 1/4 \cdot 2^2 + 1/8 \cdot 2^3 = \sum_{k=0}^{\infty} (1/2)^k \cdot 2^k = \sum_{k=0}^{\infty} 1 = \infty$$

SAST ≈ PAST

Theorem: Equivalence of $\mathtt{PAST}_{\mathbf{f}}$ and $\mathtt{SAST}_{\mathbf{f}}$

If a PTRS $\mathcal{P}$ has only finitely many rules and the corresponding signature contains a function symbol of at least arity 2, then:

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5. $\mathbb{E}(\mu_{\mathsf{c}(t,t)}) = \infty$ as before $\Rightarrow$ not PAST

Summary

- Definition and Differences between AST, PAST, and SAST

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- Modularity

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Disjoint Unions	Yes	No	Yes
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