Analyzing Almost-Sure Termination of Probabilistic Term Rewriting via Innermost Almost-Sure Termination

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Research Group Computer Science 2
"Programming Languages and Verification"

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Termination of TRSs

$\mathcal{R}_{len}$:

$$\begin{array}{rcl} \mathsf{len}(\mathsf{nil}) & \to & \mathcal{O} \\ \mathsf{len}(\mathsf{cons}(x, y)) & \to & \mathsf{s}(\mathsf{len}(y)) \end{array}$$

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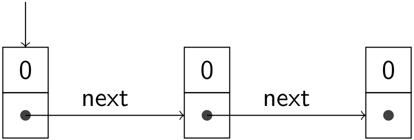

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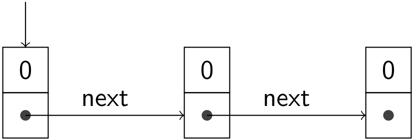

$\mathsf{len}(\mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{nil}))))$

Termination of TRSs

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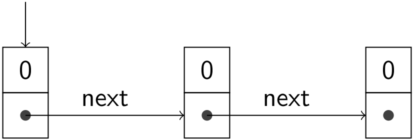

$$\mathsf{len}(\mathsf{cons}(\mathcal{O},\mathsf{cons}(\mathcal{O},\mathsf{cons}(\mathcal{O},\mathsf{nil}))))$$
$$\to_{\mathcal{R}_{len}} \mathsf{s}(\mathsf{len}(\mathsf{cons}(\mathcal{O},\mathsf{cons}(\mathcal{O},\mathsf{nil}))))$$

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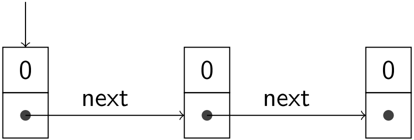

$$\begin{array}{ll} & \mathsf{len}(\mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} & \mathsf{s}(\mathsf{len}(\mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} & \mathsf{s}(\mathsf{s}(\mathsf{len}(\mathsf{cons}(\mathcal{O}, \mathsf{nil})))) \end{array}$$

Termination of TRSs

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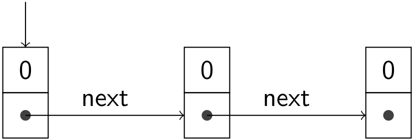

$$\begin{aligned} & \mathsf{len}(\mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} \; & \mathsf{s}(\mathsf{len}(\mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} \; & \mathsf{s}(\mathsf{s}(\mathsf{len}(\mathsf{cons}(\mathcal{O}, \mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} \; & \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{len}(\mathsf{nil})))) \end{aligned}$$

Termination of TRSs

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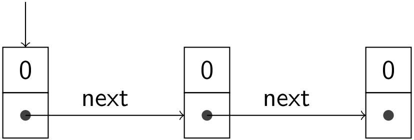

$$\begin{aligned} & \mathsf{len}(\mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} \quad & \mathsf{s}(\mathsf{len}(\mathsf{cons}(\mathcal{O}, \mathsf{cons}(\mathcal{O}, \mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} \quad & \mathsf{s}(\mathsf{s}(\mathsf{len}(\mathsf{cons}(\mathcal{O}, \mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} \quad & \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{len}(\mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} \quad & \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathcal{O}))) \end{aligned}$$

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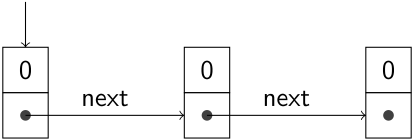

$$\begin{array}{rl} & \mathsf{len}(\mathsf{cons}(\mathcal{O},\mathsf{cons}(\mathcal{O},\mathsf{cons}(\mathcal{O},\mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} & \mathsf{s}(\mathsf{len}(\mathsf{cons}(\mathcal{O},\mathsf{cons}(\mathcal{O},\mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} & \mathsf{s}(\mathsf{s}(\mathsf{len}(\mathsf{cons}(\mathcal{O},\mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} & \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{len}(\mathsf{nil})))) \\ \to_{\mathcal{R}_{len}} & \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathcal{O}))) \end{array}$$

$\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \ldots$

Where do we use TRSs?

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- Turing complete programming language

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- Backend language of tools for the analysis of programming languages

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Java

Prolog

C

Haskell

Where do we use TRSs?

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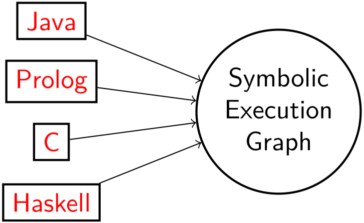

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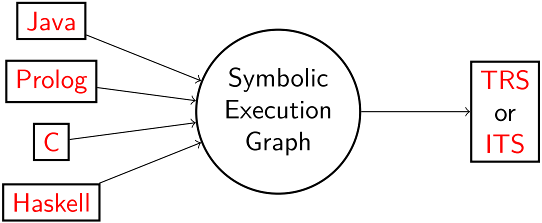

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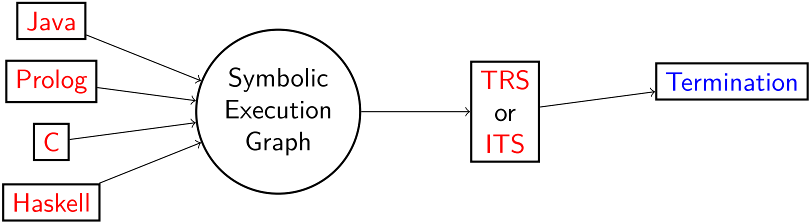

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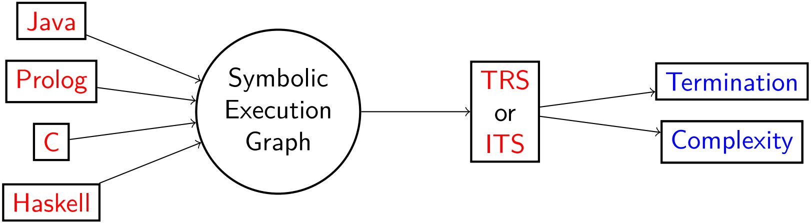

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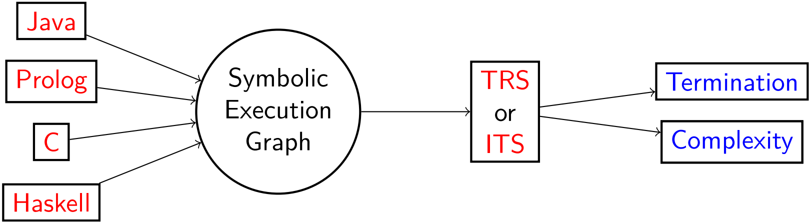

- TRS: especially good for the analysis of algorithms concerning algebraic/user-defined data structures (lists, graphs, etc.)

Non-Determinism and Evaluation Strategies

$\mathcal{R}_{plus}$:

$$\begin{aligned} \mathsf{plus}(\mathcal{O}, y) &\to y \\ \mathsf{plus}(\mathsf{s}(x), y) &\to \mathsf{s}(\mathsf{plus}(x, y)) \end{aligned}$$

Non-Determinism and Evaluation Strategies

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$$\begin{array}{rcl} \mathsf{plus}(\mathcal{O}, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

$$\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{plus}(\mathcal{O}, \mathcal{O}))$$

Non-Determinism and Evaluation Strategies

$\mathcal{R}_{plus}$:

$$\mathsf{plus}(\mathcal{O}, y) \to y$$
$$\mathsf{plus}(\mathsf{s}(x), y) \to \mathsf{s}(\mathsf{plus}(x, y))$$

$$\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{plus}(\mathcal{O}, \mathcal{O}))$$

$$\searrow$$

$$\mathsf{s}(\mathsf{plus}(\mathcal{O}, \mathsf{plus}(\mathcal{O}, \mathcal{O})))$$

Non-Determinism and Evaluation Strategies

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$$\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{plus}(\mathcal{O}, \mathcal{O}))$$

$$\downarrow$$

$$\mathsf{s}(\mathsf{plus}(\mathcal{O}, \mathsf{plus}(\mathcal{O}, \mathcal{O})))$$

$$\downarrow$$

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Non-Determinism and Evaluation Strategies

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$$\mathsf{plus}(\mathsf{s}(\mathcal{O}), \mathsf{plus}(\mathcal{O}, \mathcal{O}))$$

$$\to \mathsf{s}(\mathsf{plus}(\mathcal{O}, \mathsf{plus}(\mathcal{O}, \mathcal{O})))$$

$$\to \mathsf{s}(\mathsf{plus}(\mathcal{O}, \mathcal{O}))$$

$$\to \mathsf{s}(\mathcal{O})$$

Non-Determinism and Evaluation Strategies

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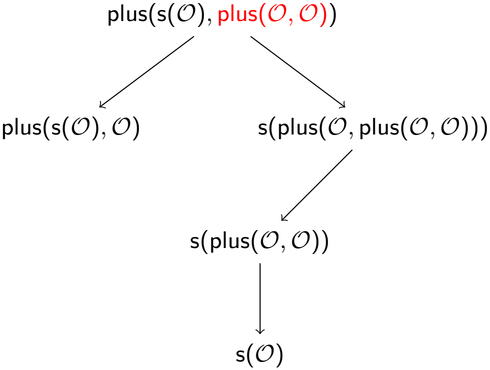

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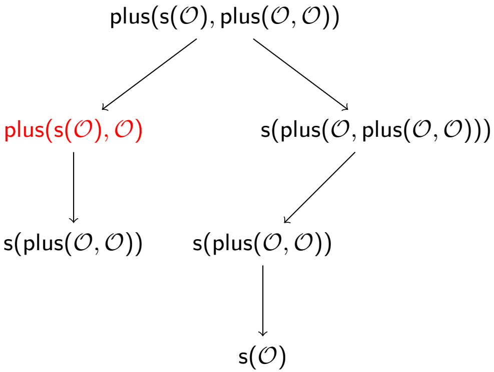

Non-Determinism and Evaluation Strategies

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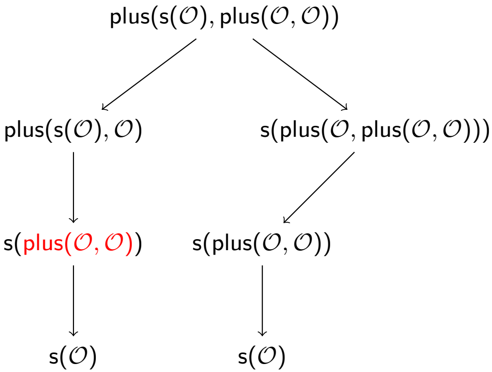

Non-Determinism and Evaluation Strategies

$$
\mathcal{R}_{plus}: \quad
\begin{aligned}
\mathsf{plus}(\mathcal{O}, y) &\to y \\
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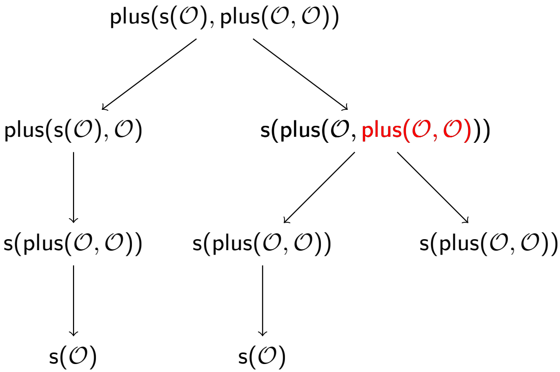
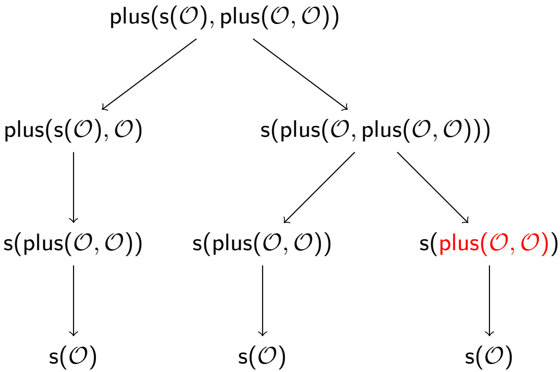
\end{aligned}
$$



Non-Determinism and Evaluation Strategies

$\mathcal{R}_{plus}$:

$$\begin{aligned} \mathsf{plus}(\mathcal{O}, y) &\rightarrow y \\ \mathsf{plus}(\mathsf{s}(x), y) &\rightarrow \mathsf{s}(\mathsf{plus}(x, y)) \end{aligned}$$



Non-Determinism and Evaluation Strategies

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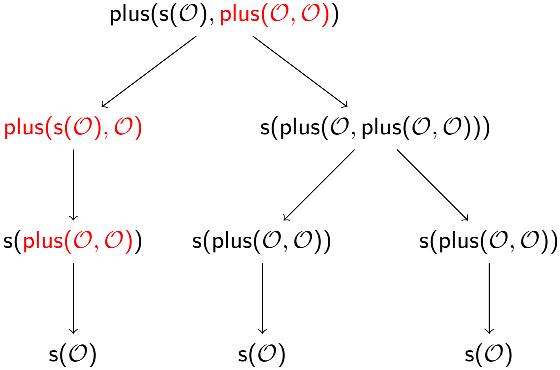

Innermost evaluation: always use an innermost reducible expression

Innermost Termination vs. Termination

$\mathcal{R}_1$:

$$\begin{array}{rcl} \mathsf{f}(\mathsf{a},\mathsf{b},x) & \to & \mathsf{f}(x,x,x) \\ \mathsf{g}(x,y) & \to & x \\ \mathsf{g}(x,y) & \to & y \end{array}$$

Innermost Termination vs. Termination

$\mathcal{R}_1$:

$$\begin{aligned} \mathsf{f}(\mathsf{a}, \mathsf{b}, x) &\to \mathsf{f}(x, x, x) \\ \mathsf{g}(x, y) &\to x \\ \mathsf{g}(x, y) &\to y \end{aligned}$$

Not Terminating:

Innermost Termination vs. Termination

$\mathcal{R}_1$:

$$
\begin{aligned}
\mathsf{f}(\mathsf{a},\mathsf{b},x) &\to \mathsf{f}(x,x,x)\\
\mathsf{g}(x,y) &\to x\\
\mathsf{g}(x,y) &\to y
\end{aligned}
$$

Not Terminating:

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b}))$$

Innermost Termination vs. Termination

$\mathcal{R}_1$:

$$\begin{array}{rcl} \mathsf{f}(\mathsf{a},\mathsf{b},x) & \to & \mathsf{f}(x,x,x) \\ \mathsf{g}(x,y) & \to & x \\ \mathsf{g}(x,y) & \to & y \end{array}$$

Not Terminating:

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}))$$

Innermost Termination vs. Termination

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$$\begin{array}{rcl} \mathsf{f}(\mathsf{a},\mathsf{b},x) & \to & \mathsf{f}(x,x,x) \\ \mathsf{g}(x,y) & \to & x \\ \mathsf{g}(x,y) & \to & y \end{array}$$

Not Terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \end{aligned}$$

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$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \end{aligned}$$

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Not Terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \cdots \end{aligned}$$

Innermost Termination vs. Termination

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$$\begin{array}{rcl} \mathsf{f}(\mathsf{a},\mathsf{b},x) & \to & \mathsf{f}(x,x,x) \\ \mathsf{g}(x,y) & \to & x \\ \mathsf{g}(x,y) & \to & y \end{array}$$

Not Terminating:

$$\begin{array}{rl} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) & \to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ & \to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ & \to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \\ & \to_{\mathcal{R}_1} \cdots \end{array}$$

But **Innermost Terminating:**

Innermost Termination vs. Termination

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Not Terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \dots \end{aligned}$$

But **Innermost Terminating:**

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b}))$$

Innermost Termination vs. Termination

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Not Terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \cdots \end{aligned}$$

But **Innermost Terminating:**

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \xrightarrow{\mathrm{i}}_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{a})$$

Innermost Termination vs. Termination

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$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},x) &\to \mathsf{f}(x,x,x) \\ \mathsf{g}(x,y) &\to x \\ \mathsf{g}(x,y) &\to y \end{aligned}$$

Not Terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \dots \end{aligned}$$

But **Innermost Terminating:**

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \xrightarrow{\mathsf{i}}_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{a}) \xrightarrow{\mathsf{i}}_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{a},\mathsf{a})$$

Innermost Termination vs. Termination

$\mathcal{R}_1$:
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Not Terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \cdots \end{aligned}$$

But **Innermost Terminating:**

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \xrightarrow{\mathsf{i}}_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{a}) \xrightarrow{\mathsf{i}}_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{a},\mathsf{a}) \leftarrow \text{normal form}$$

Innermost Termination vs. Termination

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Not Terminating:

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$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \xrightarrow{\mathsf{i}}_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{a}) \xrightarrow{\mathsf{i}}_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{a},\mathsf{a}) \leftarrow \text{normal form}$$

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b}))$$

Innermost Termination vs. Termination

$\mathcal{R}_1$:
$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},x) &\to \mathsf{f}(x,x,x) \\ \mathsf{g}(x,y) &\to x \\ \mathsf{g}(x,y) &\to y \end{aligned}$$

Not Terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \cdots \end{aligned}$$

But **Innermost Terminating:**

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \xrightarrow{\mathsf{i}}_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{a}) \xrightarrow{\mathsf{i}}_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{a},\mathsf{a}) \leftarrow \text{normal form}$$

$$\mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \xrightarrow{\mathsf{i}}_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{b})$$

Innermost Termination vs. Termination

$\mathcal{R}_1$:

$$\begin{array}{rcl} f(a,b,x) & \to & f(x,x,x) \\ g(x,y) & \to & x \\ g(x,y) & \to & y \end{array}$$

Not Terminating:

$$\begin{array}{rl} f(a,b,g(a,b)) & \to_{\mathcal{R}_1} f(g(a,b),g(a,b),g(a,b)) \\ & \to_{\mathcal{R}_1} f(a,g(a,b),g(a,b)) \\ & \to_{\mathcal{R}_1} f(a,b,g(a,b)) \\ & \to_{\mathcal{R}_1} \dots \end{array}$$

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Innermost Termination vs. Termination

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Not Terminating:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{g}(\mathsf{a},\mathsf{b}),\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \mathsf{f}(\mathsf{a},\mathsf{b},\mathsf{g}(\mathsf{a},\mathsf{b})) \\ &\to_{\mathcal{R}_1} \dots \end{aligned}$$

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Innermost Termination vs. (full) Termination

$\mathcal{R}_2$:

$$\begin{array}{rcl} \mathsf{f}(\mathsf{a}) & \to & \mathsf{f}(\mathsf{a}) \\ \mathsf{a} & \to & \mathsf{b} \end{array}$$

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Not Terminating:

Innermost Termination vs. (full) Termination

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Not Terminating:

$$\mathsf{f}(\mathsf{a})$$

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$$\begin{aligned} \mathsf{f}(\mathsf{a}) &\to \mathsf{f}(\mathsf{a}) \\ \mathsf{a} &\to \mathsf{b} \end{aligned}$$

Not Terminating:

$$\mathsf{f}(\mathsf{a}) \to_{\mathcal{R}_2} \mathsf{f}(\mathsf{a})$$

Innermost Termination vs. (full) Termination

$\mathcal{R}_2$:

$$\begin{aligned} f(a) &\to f(a) \\ a &\to b \end{aligned}$$

Not Terminating:

$$f(a) \to_{\mathcal{R}_2} f(a) \to_{\mathcal{R}_2} \ldots$$

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Innermost Termination vs. (full) Termination

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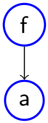

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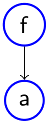

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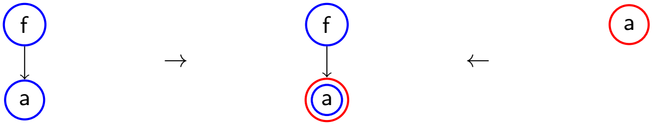

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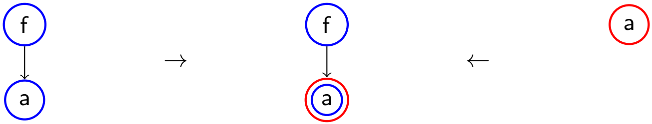

mgu $\sigma = \varnothing = id$

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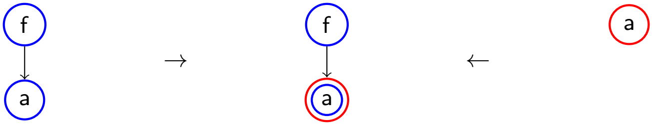

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$\to \mathcal{R}_2$ is overlapping.

Overlapping cont.

$\mathcal{R}_1$:

$$\begin{aligned} \mathsf{f}(\mathsf{a},\mathsf{b},x) &\to \mathsf{f}(x,x,x) \\ \mathsf{g}(x,y) &\to x \\ \mathsf{g}(x,y) &\to y \end{aligned}$$

Overlapping cont.

$\mathcal{R}_1$:

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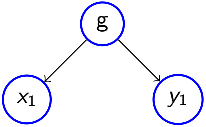

Overlapping cont.

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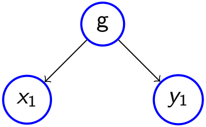
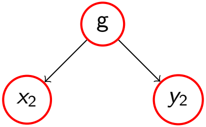
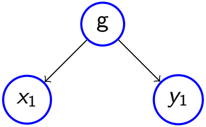




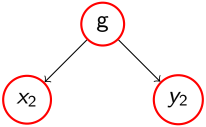
Overlapping cont.

$\mathcal{R}_1$:

$$\begin{aligned} f(a,b,x) &\to f(x,x,x) \\ g(x,y) &\to x \\ g(x,y) &\to y \end{aligned}$$



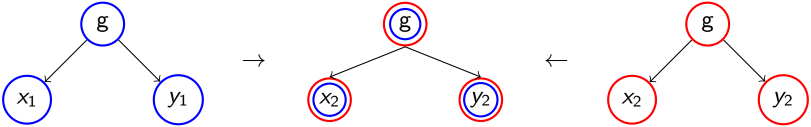
$\to$ $\leftarrow$



Overlapping cont.

$\mathcal{R}_1$:

$$
\begin{aligned}
\mathsf{f}(\mathsf{a},\mathsf{b},x) &\to \mathsf{f}(x,x,x)\\
\color{blue}{\mathsf{g}(x,y)} &\to x\\
\color{red}{\mathsf{g}(x,y)} &\to y
\end{aligned}
$$



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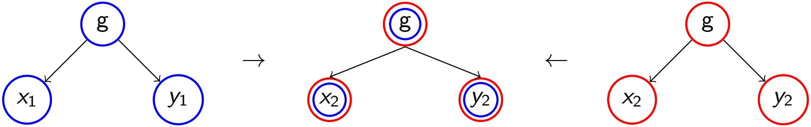

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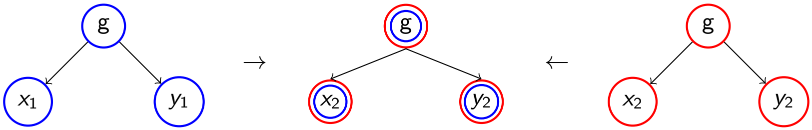

mgu $\sigma = \{x_1/x_2, y_1/y_2\}$

$\to \mathcal{R}_1$ is overlapping.

Overlapping cont.

non-overlapping (Idea)

In a term at a certain position can only be a single rewrite step possible and rewriting at another position does not interfere with this possibility

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$$\text{plus}(\text{s}(\mathcal{O}), \text{plus}(\mathcal{O}, \mathcal{O}))$$

Overlapping cont.

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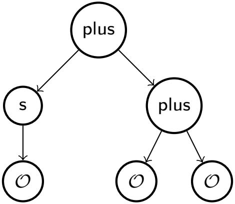
$\rightarrow$ $\mathcal{R}_{plus}$ is non-overlapping.



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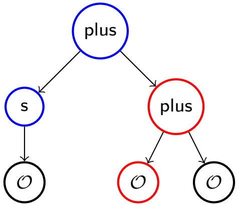

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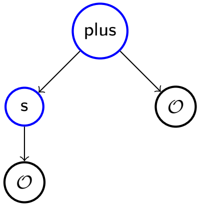

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Theorem

If $\mathcal{R}$ is non-overlapping then:

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$\Rightarrow$ there exists a step $t_N \to_{\mathcal{R}} t_{N+1}$ that is not an innermost step ($t_N \stackrel{i}{\not\to}_{\mathcal{R}} t_{N+1}$) performed at some position π
Due to minimality rewriting $t_N \stackrel{\tau}{\to}_{\mathcal{R}} s_N$ at a position τ below π can not start an infinite evaluation

Proof cont.

From

$$t_N \to_{\mathcal{R}} t_{N+1} \to_{\mathcal{R}} t_{N+2} \to_{\mathcal{R}} \cdots$$

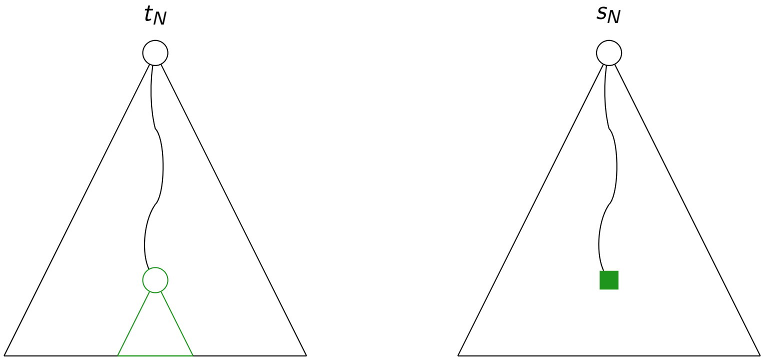
construct infinite evaluation

$$t_N \xrightarrow{\tau}_{\mathcal{R}} s_N \to_{\mathcal{R}} s_{N+1} \to_{\mathcal{R}} s_{N+2} \to_{\mathcal{R}} \cdots$$

Proof cont.

Rewriting possibilities:

1. Rewriting at an orthogonal position
2. Rewriting at a position above
3. Rewriting the colored position



Proof cont.

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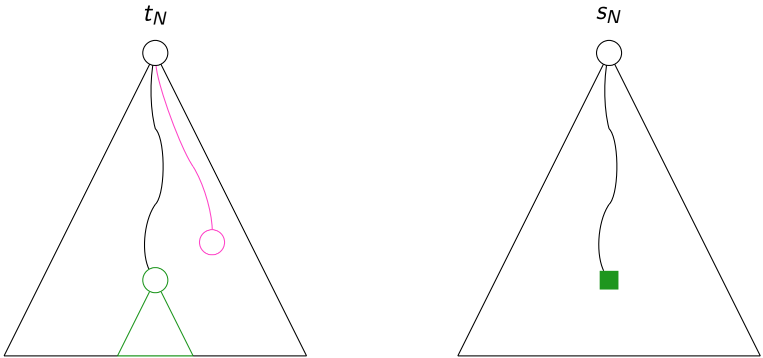

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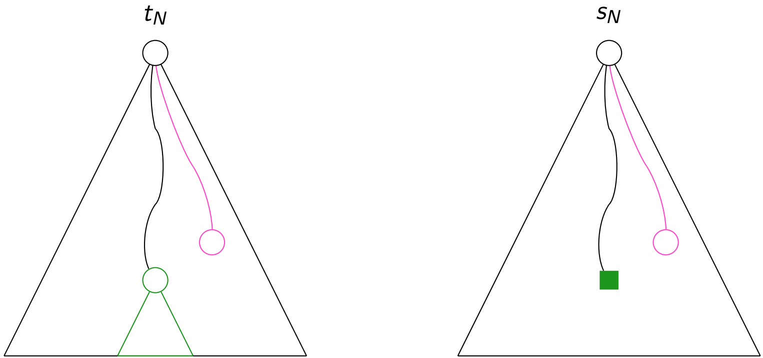

Proof cont.

Rewriting possibilities:

1. Rewriting at an orthogonal position
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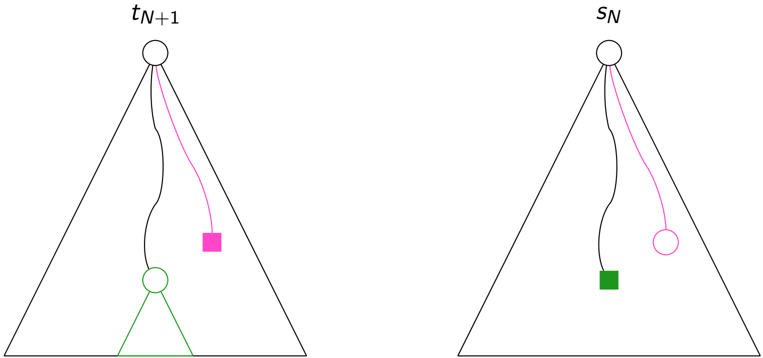

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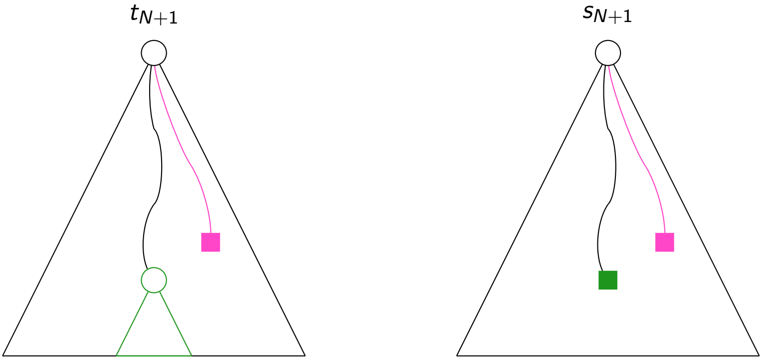

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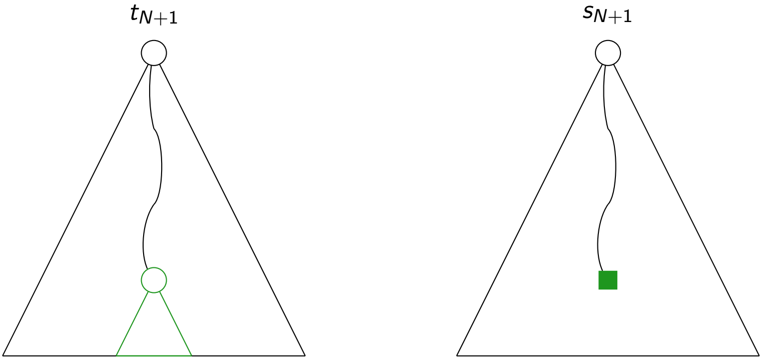

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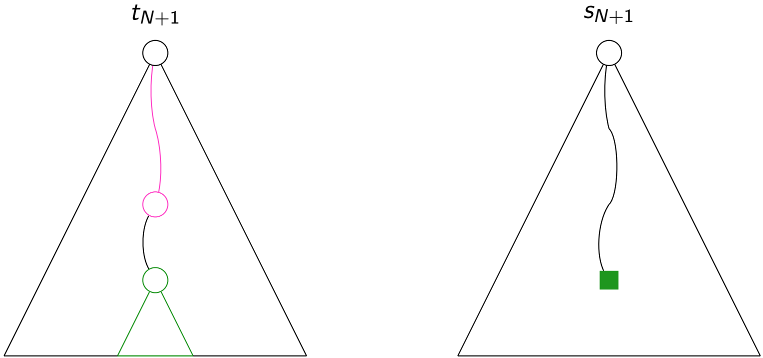

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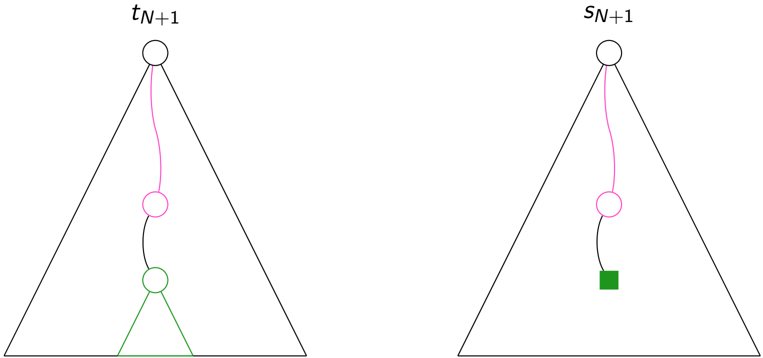

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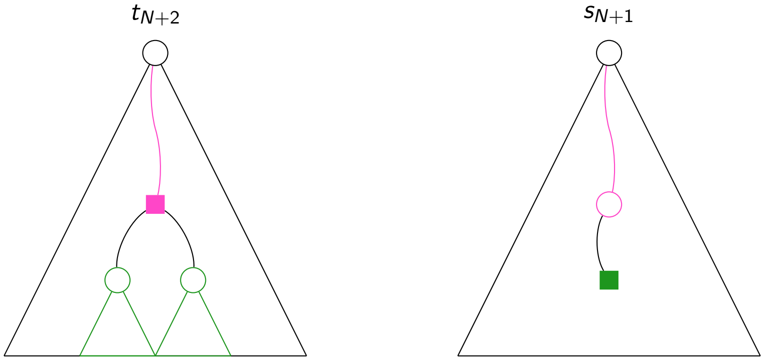

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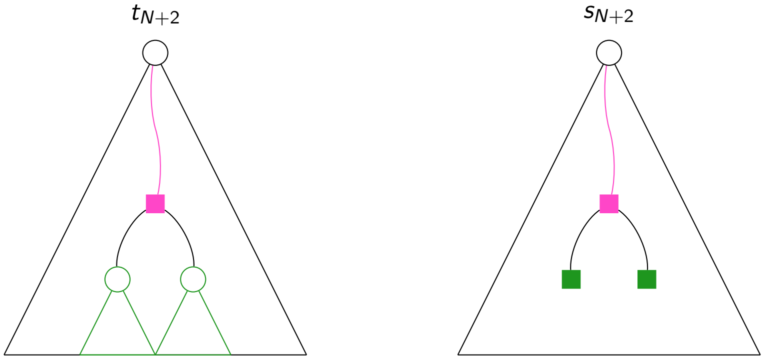

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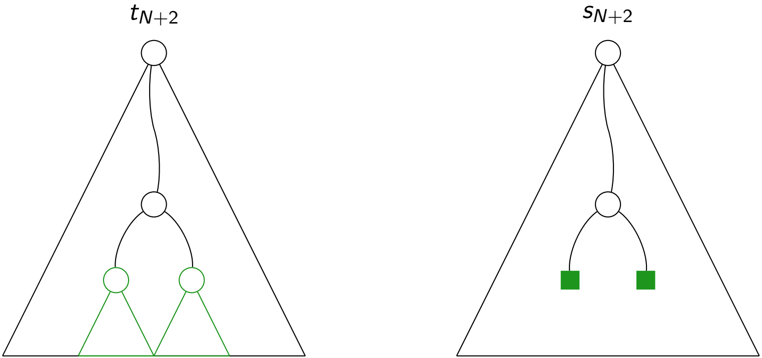
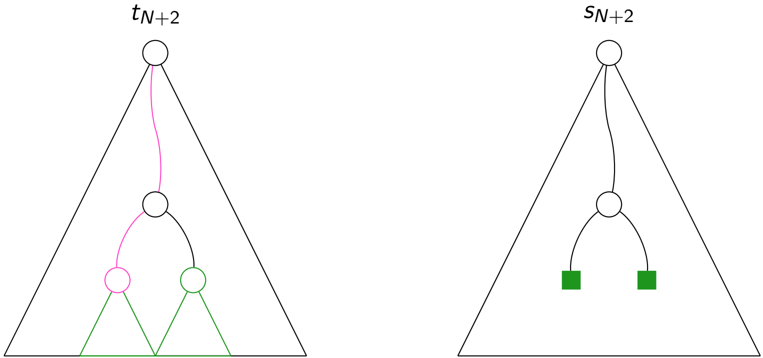


Proof cont.

Rewriting possibilities:

1. Rewriting at an orthogonal position
2. Rewriting at a position above
3. Rewriting the colored position ← finite direct repititions



Proof cont.

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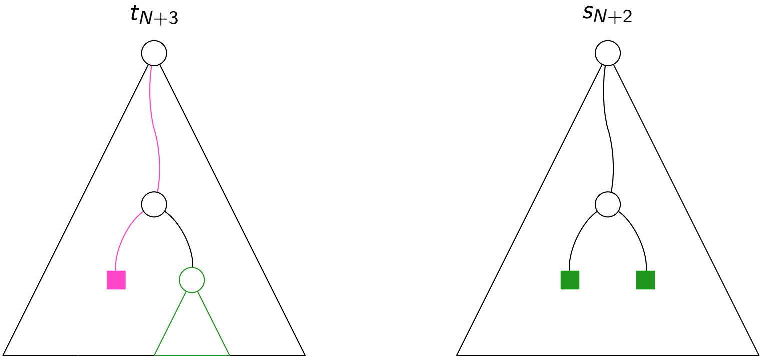

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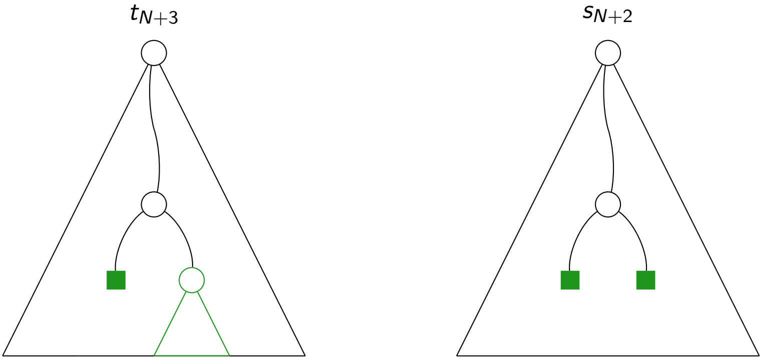

Termination of PTRSs

$$\mathcal{S}_{len}: \quad \begin{aligned} \mathsf{len}(\mathsf{nil}) &\to \{{}^1\!/_2 : \mathcal{O}, {}^1\!/_2 : \mathsf{len}(\mathsf{nil})\} \\ \mathsf{len}(\mathsf{cons}(x, y)) &\to \{{}^1\!/_2 : \mathsf{s}(\mathsf{len}(y)), {}^1\!/_2 : \mathsf{len}(\mathsf{cons}(x, y))\} \end{aligned}$$

Termination of PTRSs

$\mathcal{S}_{len}$:

$$\mathsf{len}(\mathsf{nil}) \rightarrow \{1/2 : \mathcal{O}, 1/2 : \mathsf{len}(\mathsf{nil})\}$$
$$\mathsf{len}(\mathsf{cons}(x, y)) \rightarrow \{1/2 : \mathsf{s}(\mathsf{len}(y)), 1/2 : \mathsf{len}(\mathsf{cons}(x, y))\}$$

Computation of $\mathsf{len}(\mathsf{nil})$:

Termination of PTRSs

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Termination of PTRSs

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Computation of len(nil):

$$\begin{aligned} & \{1 : \mathsf{len}(\mathsf{nil})\} \\ \rightrightarrows_{\mathcal{S}_{len}} \quad & \{1/2 : \mathsf{len}(\mathsf{nil}), 1/2 : \mathcal{O}\} \end{aligned}$$

Termination of PTRSs

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Termination of PTRSs

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Termination of PTRSs

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- $\mathcal{S}_{len}$ is AST

Innermost AST vs. (full) AST

$\mathcal{S}_2$:

$$\begin{aligned} \mathsf{f}(\mathsf{a}) &\to \{1 : \mathsf{f}(\mathsf{a})\} \\ \mathsf{a} &\to \{1 : \mathsf{b}\} \end{aligned}$$

Innermost AST vs. (full) AST

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Not AST:

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Not AST:

$$\{1 : \mathsf{f}(\mathsf{a})\} \rightrightarrows_{\mathcal{S}_2} \{1 : \mathsf{f}(\mathsf{a})\}$$

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Innermost AST vs. (full) AST

$\mathcal{S}_2$:
$$\begin{aligned} \mathsf{f}(\mathsf{a}) &\to \{1 : \mathsf{f}(\mathsf{a})\} \\ \mathsf{a} &\to \{1 : \mathsf{b}\} \end{aligned}$$

Not AST:

$$\{1 : \mathsf{f}(\mathsf{a})\} \rightrightarrows_{\mathcal{S}_2} \{1 : \mathsf{f}(\mathsf{a})\} \rightrightarrows_{\mathcal{S}_2} \dots$$

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$$\{1 : \mathsf{f}(\mathsf{a})\} \overset{\mathsf{i}}{\rightrightarrows}_{\mathcal{S}_2} \{1 : \mathsf{f}(\mathsf{b})\} \leftarrow \text{normal form}$$

Properties for Equality of Termination

Does non-overlapping still suffice?

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Not AST:

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Not AST: directly performing the f-rule to duplicate the g's results in

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Not AST: directly performing the f-rule to duplicate the g's results in

$\mathcal{S}_3$:

$$\mathsf{g} \rightarrow \{{}^3\!/\!{}_4 : \mathsf{c}(\mathsf{g}, \mathsf{g}), {}^1\!/\!{}_4 : \bot\}$$

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$\mathcal{S}_3$:
$$\mathsf{g} \to \{3/4 : \mathsf{c}(\mathsf{g}, \mathsf{g}), 1/4 : \bot\}$$

$\to$ Biased random walk with $p = 3/4 > 1/2$, hence not AST.

Properties for Equality of Termination cont.

But **Innermost AST:**

$$\mathsf{g} \to \{{}^{3}\!/\!_{4} : \mathsf{f}(\mathsf{g}), {}^{1}\!/\!_{4} : \bot\}$$
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Properties for Equality of Termination cont.

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μ_0 : $\boxed{1 : \mathsf{g}}$

Properties for Equality of Termination cont.

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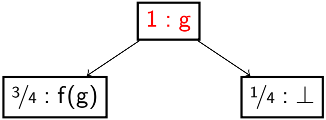

Properties for Equality of Termination cont.

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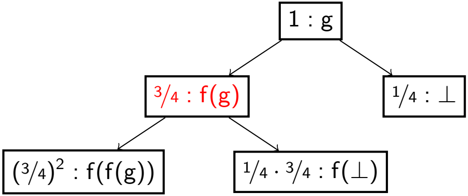
μ_1 :

μ_2 :



Properties for Equality of Termination cont.

But **Innermost AST:**

$\mathsf{g} \to \{\nicefrac{3}{4} : \mathsf{f}(\mathsf{g}), \nicefrac{1}{4} : \bot\}$

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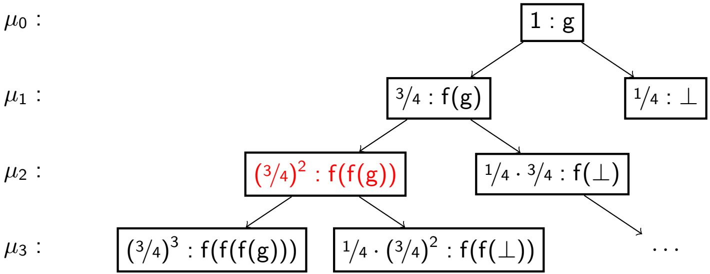

Properties for Equality of Termination cont.

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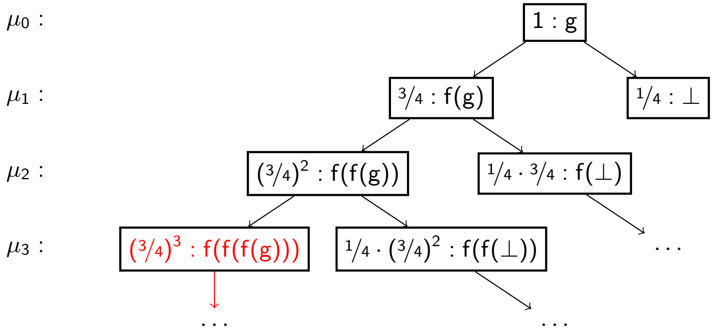

Properties for Equality of Termination cont.

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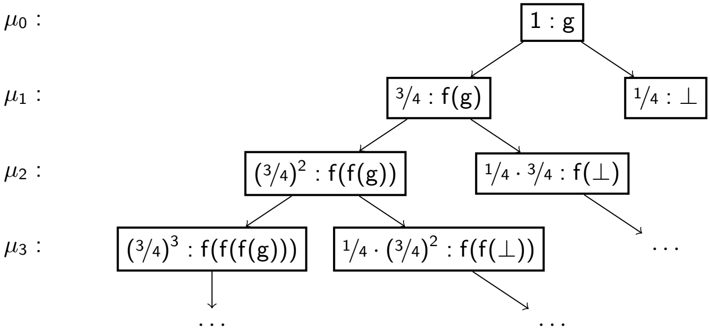

$$\lim_{k \to \infty} |\mu_k| =$$

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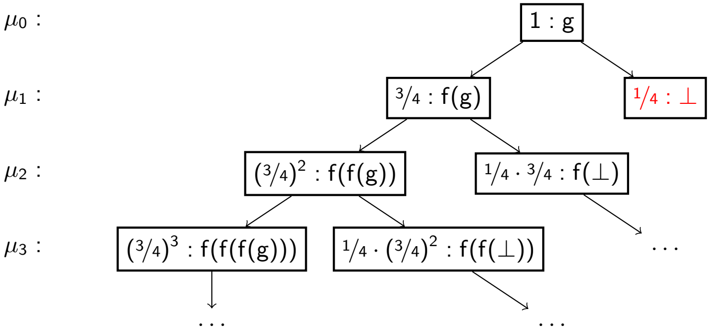

$$\lim_{k\to\infty} |\mu_k| = 1/4$$

Properties for Equality of Termination cont.

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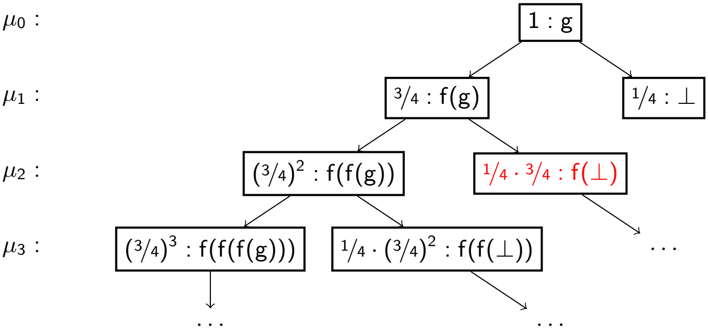

$$\lim_{k\to\infty} |\mu_k| = {}^1\!/\!_4 + {}^1\!/\!_4 \cdot {}^3\!/\!_4$$

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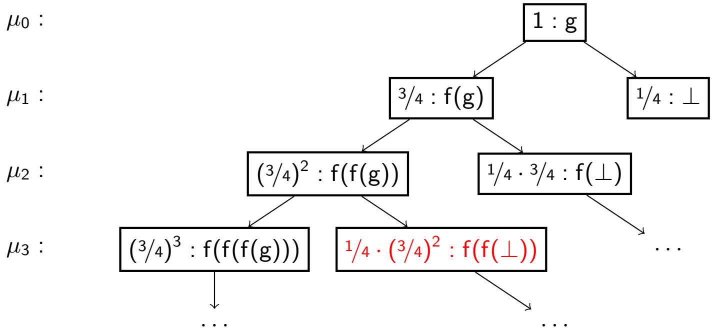

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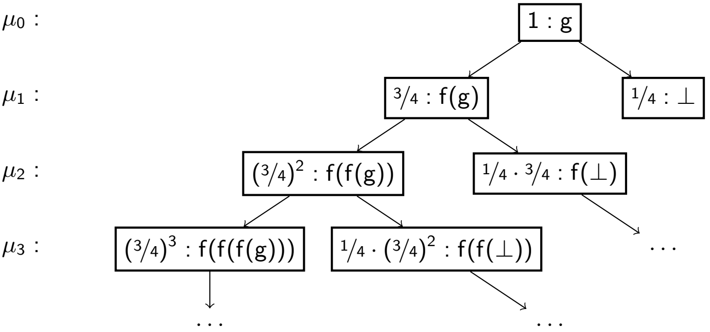

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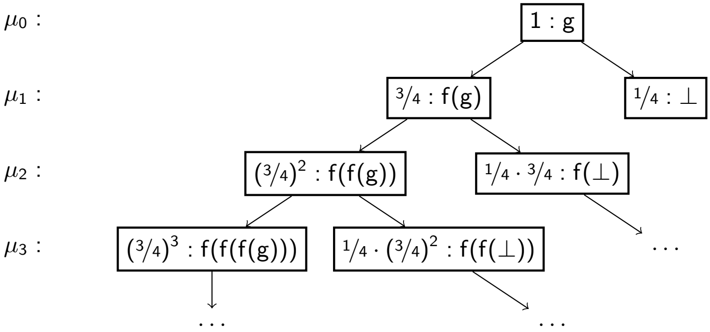

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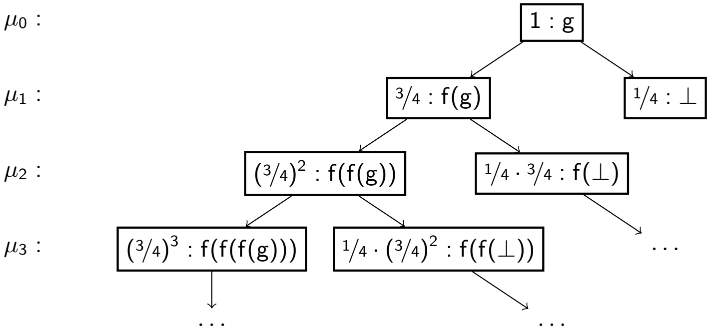

$$\lim_{k\to\infty} |\mu_k| = 1/4 + 1/4 \cdot 3/4 + 1/4 \cdot (3/4)^2 + \ldots = \sum_{i=0}^{\infty} 1/4 \cdot (3/4)^i = 1$$

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$\to \mathcal{S}_2$ is duplicating.

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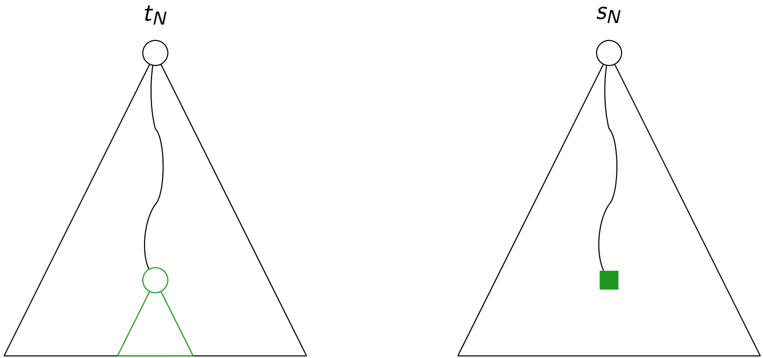
3. Calculating the probability of termination

Proof cont.

Rewriting possibilities:

1. Rewriting at an orthogonal position
2. Rewriting at a position above
3. Rewriting the colored position



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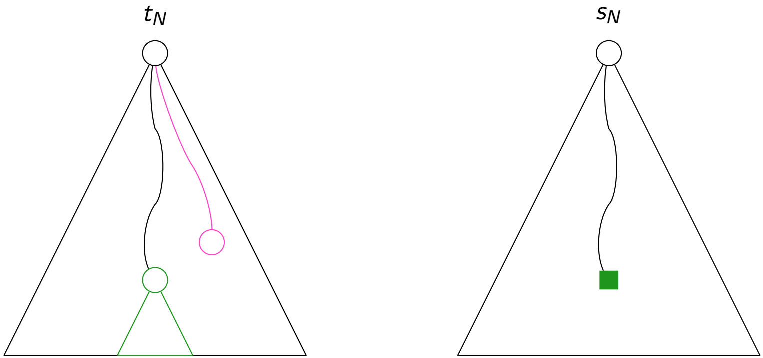

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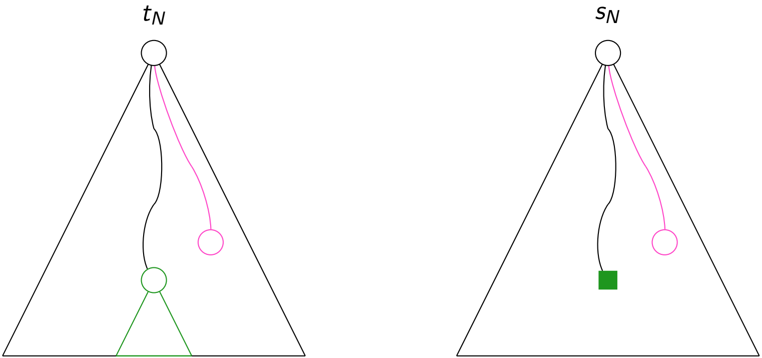

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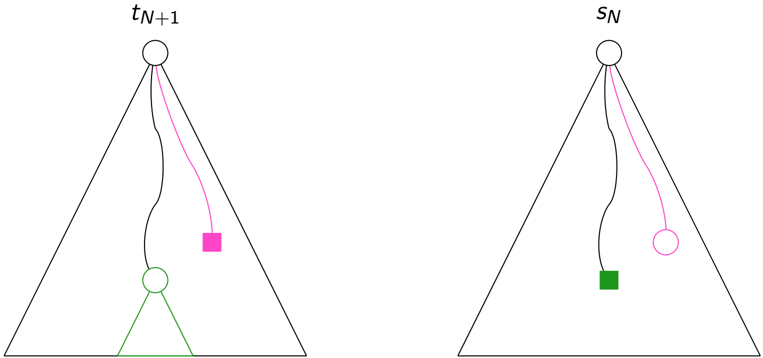

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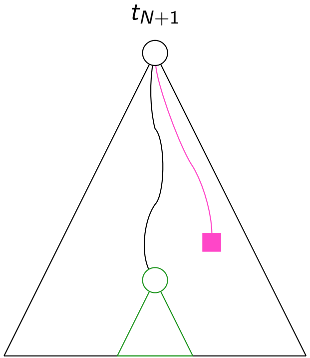

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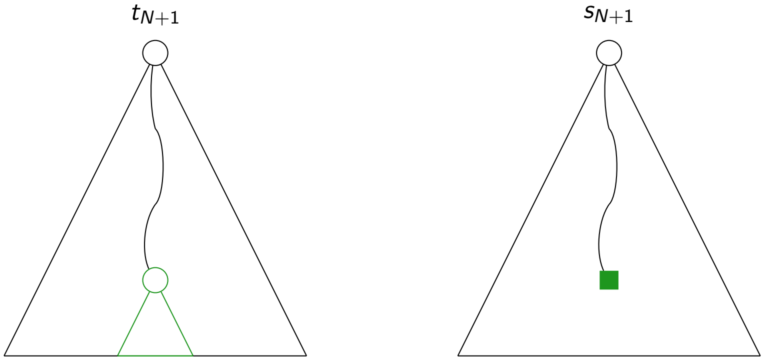

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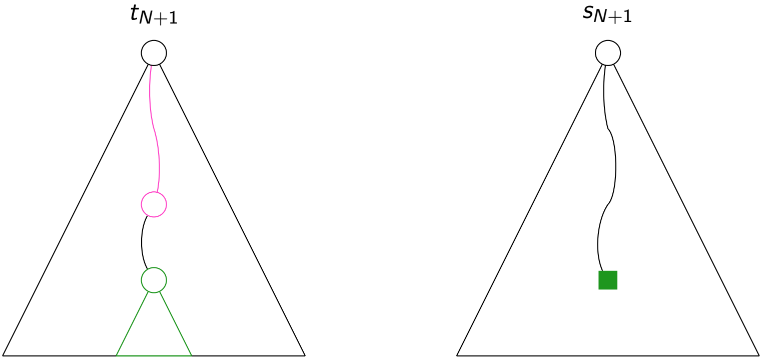

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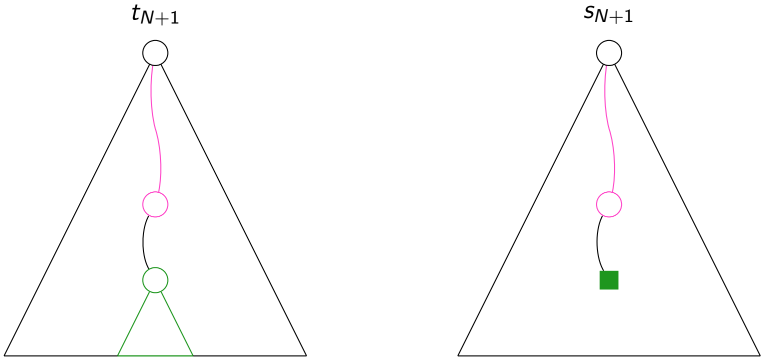

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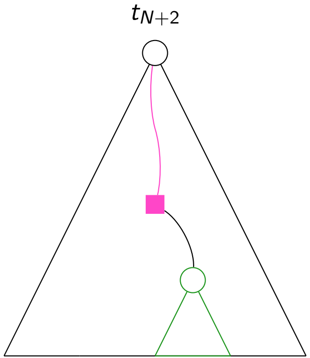

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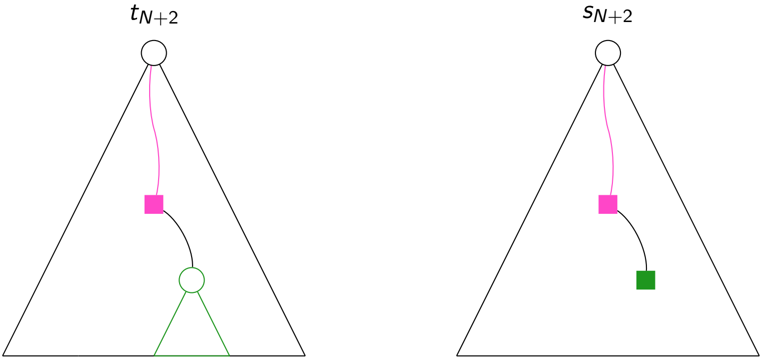

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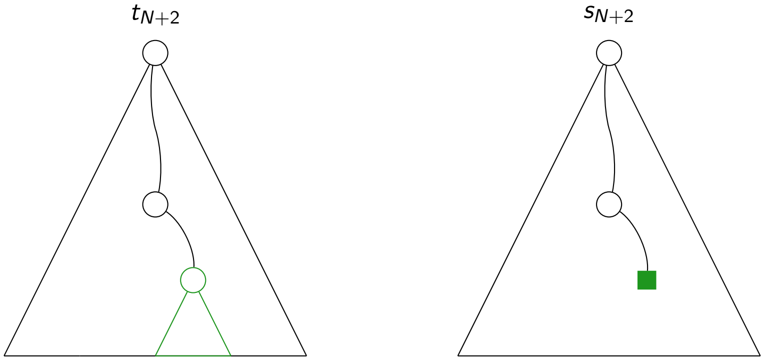

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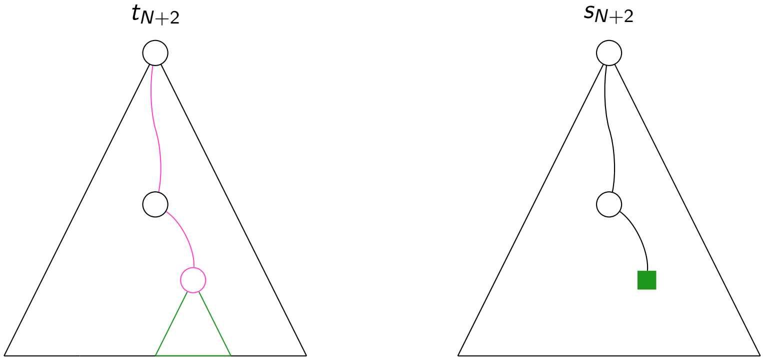

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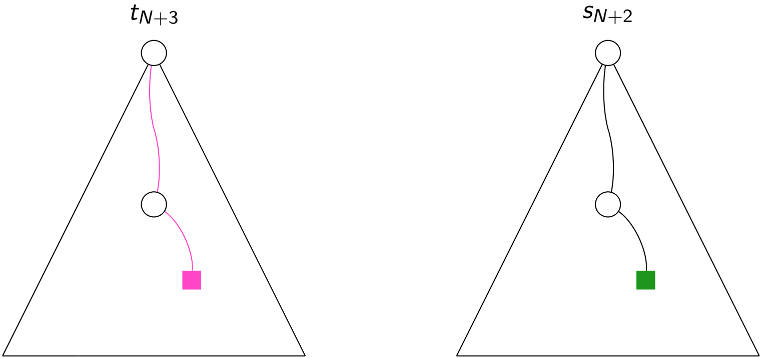

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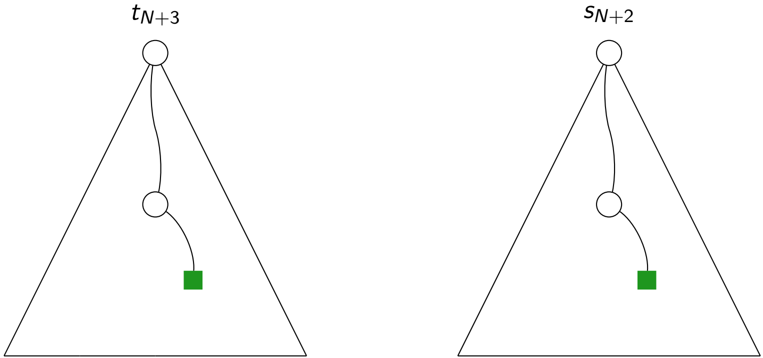

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