Reachability Analysis in Probabilistic Term Rewriting

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Reachability in Term Rewriting

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Reachability Problem

Can a target t (e.g., an error state) be reached from a start term s?

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- closely related to **confluence analysis**

From Qualitative to Quantitative Reachability

What if the program can make **random choices** during execution?
Or if the data contains uncertainty, e.g., **random noise**?

Term Rewriting

qualitative analysis

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$s \to_{\mathcal{R}}^{*} t$ ✓ / ✗

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With which probability is t reached from s?

$\mathbb{P}(s \to_{\mathcal{P}}^{*} t) \in [0, 1]$

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Research Goal

Lift reachability analysis to **probabilistic** TRSs (PTRSs) and compute **upper bounds** p:

$p < 1$ disproves *almost-sure* reachability $p = 0$ disproves reachability at all

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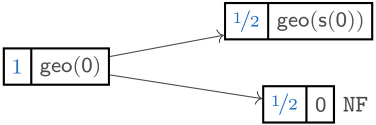

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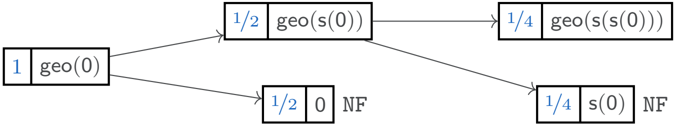

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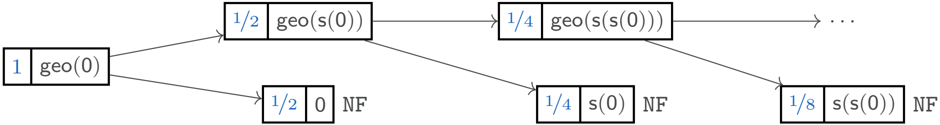

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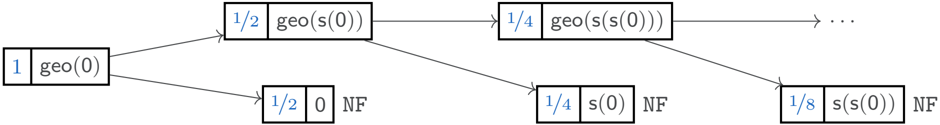

$\mathsf{s}^k(0)$ is reached from $\mathsf{geo}(0)$ with probability $(1/2)^{k+1}$ (a geometric distribution)

Maximal Reachability Probability

Collect the probability mass of t

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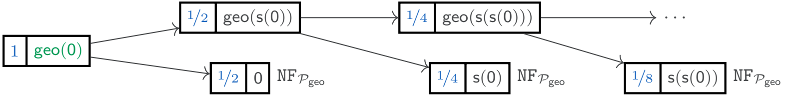

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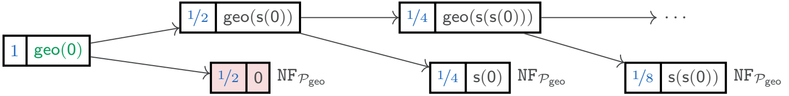

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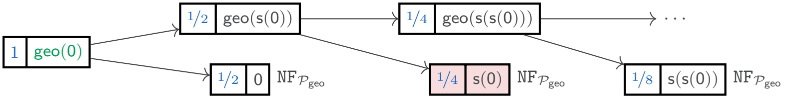

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Target $\mathsf{s}(0)$: one leaf, probability $1/4$ $\Longrightarrow$ $\mathsf{geo}(0) \dashrightarrow_{\leq 1/4}^{\mathcal{P}_{\mathsf{geo}}} \mathsf{s}(0)$

Two Techniques for Upper Bounds

Goal: compute upper bounds $s \dashrightarrow^{\mathcal{P}}_{\leq p} t$ **automatically**

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over-approximate $\to_{\mathcal{P}}$ by a **finite** MDP
read off bounds with model checkers

2. Interpretations

weight functions as **ranking functions**
bounds via the Stopping Theorem

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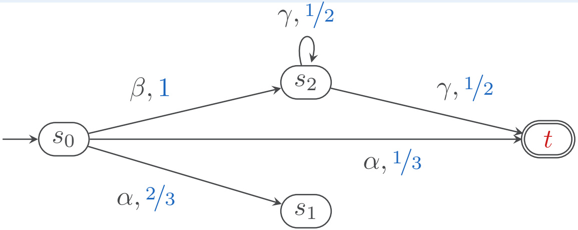

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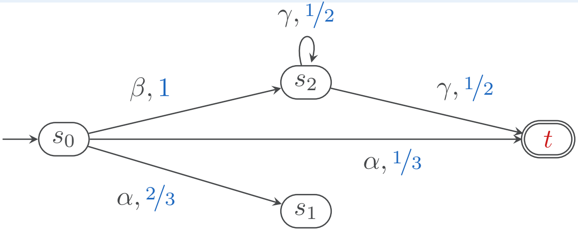

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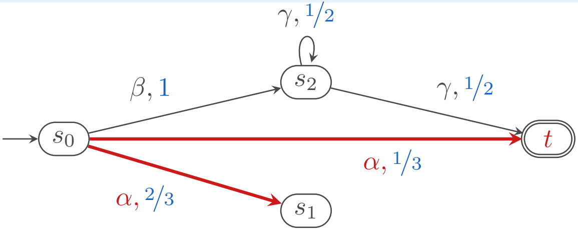

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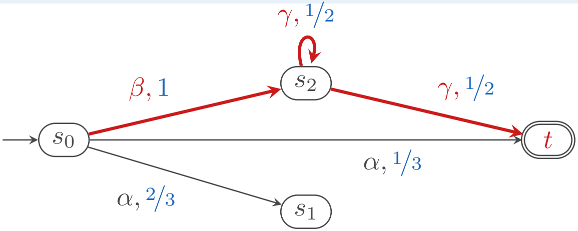

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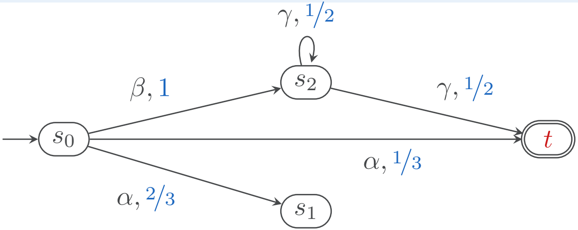

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- $\mathbb{P}^{\max}_{\mathcal{M}}(s_0, \{t\}) = \max\{1/3, 1\} = 1$

The n-Cut MDP $\mathcal{M}_n(\mathcal{P})$ (lifting [Sternagel&Yamada'19])

$\mathcal{P}_{\mathsf{geo}}$: $\chi: \ \mathsf{geo}(x) \to \{{}^{1}/{}_{2}\,\mathsf{geo}(\mathsf{s}(x)),\ {}^{1}/{}_{2}\,x\}$ start: $\mathsf{geo}(0)$, target: $\mathsf{s}(0)$

$\to_{\mathcal{P}}$ has **infinitely** many states (terms) $\rightsquigarrow$ over-approximate it by the **finite** MDP $\mathcal{M}_n(\mathcal{P})$, $n \in \mathbb{N}$

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$\downarrow$

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$$\boxed{\mathsf{geo}(0)}$$

every edge applies (ε, χ)

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- **transitions**: narrowing + cut

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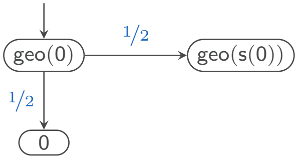

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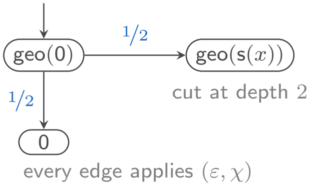

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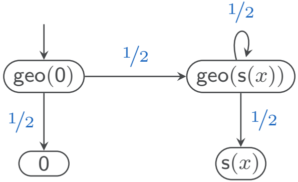

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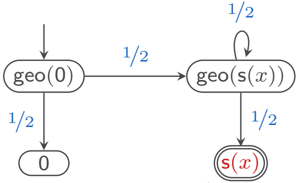

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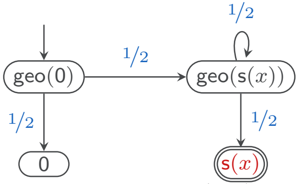

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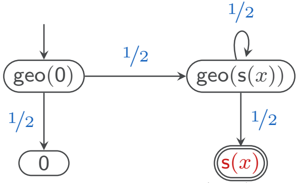

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over-approximates $1/4$, because $\mathsf{geo}(\mathsf{s}(x))$ merges all $\mathsf{geo}(\mathsf{s}^k(0))$!

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$$\downarrow$$

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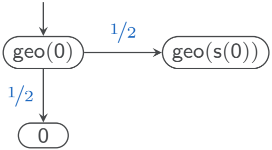

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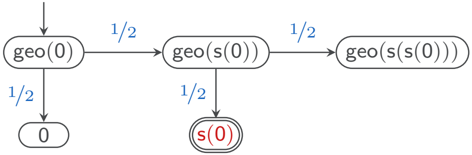

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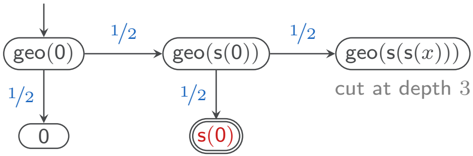

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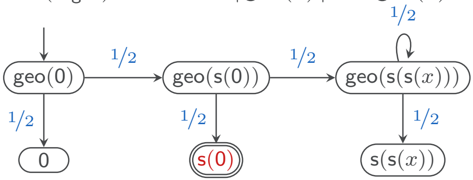

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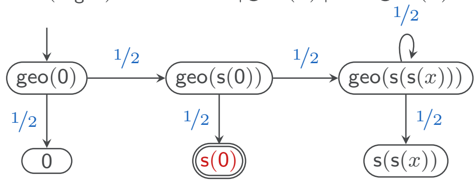

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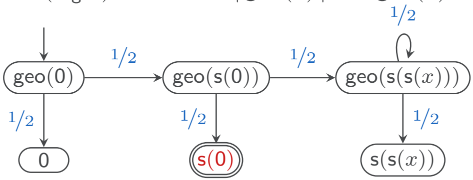

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Soundness & Refinement

$\mathbb{P}^{\max}_{\mathcal{M}_n(\mathcal{P})}(\lceil s \rceil_n, T_t^n) \leq p \implies \sup_\sigma \mathbb{P}^{\max}(s\sigma \to^*_{\mathcal{P}} t\sigma) \leq p$, and larger n yields **tighter** bounds.

Two Techniques for Upper Bounds

Goal: compute upper bounds $s \dashrightarrow^{\mathcal{P}}_{\leq p} t$ **automatically**

1. Markov Decision Processes ✓

over-approximate $\to_{\mathcal{P}}$ by a **finite** MDP

read off bounds with model checkers

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weight functions as **ranking functions**

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$$W(\mathsf{s}(x)) = x + 1 = \underbrace{{}^{1}/_{2} \cdot x + {}^{1}/_{2} \cdot (x+2)}_{\mathbb{E}_W(\{{}^{1/2}x,\ {}^{1/2}\mathsf{s}^2(x)\})} = x + 1 \quad ✓$$

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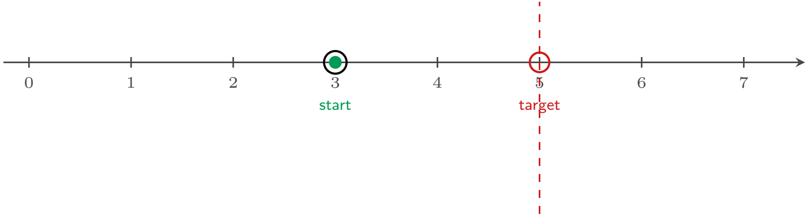

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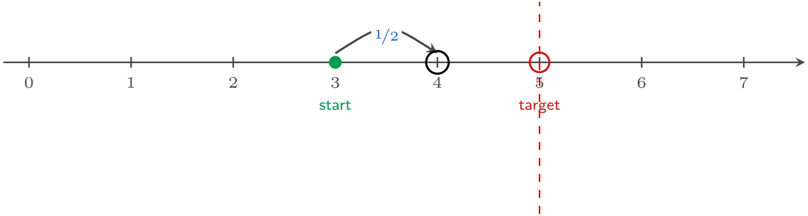

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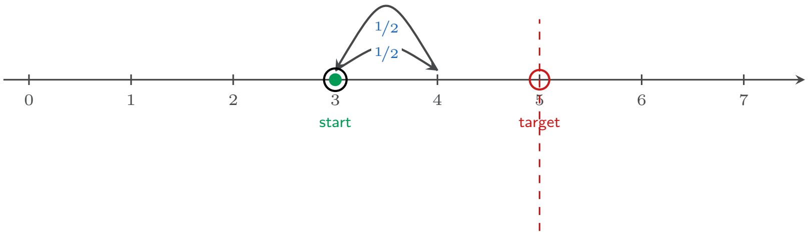

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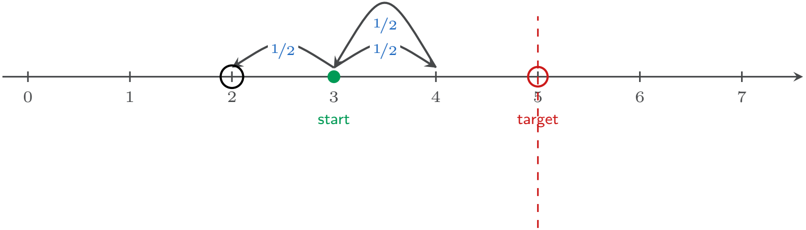

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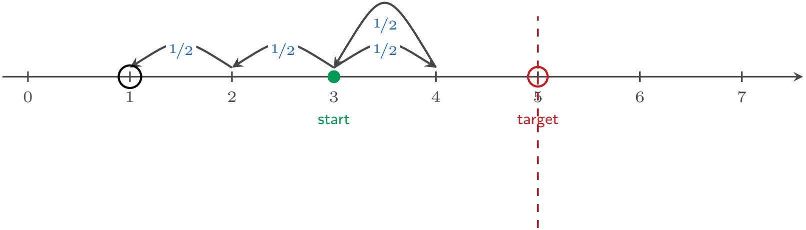

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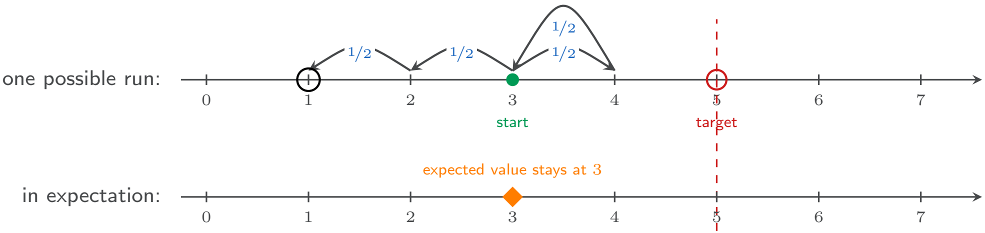

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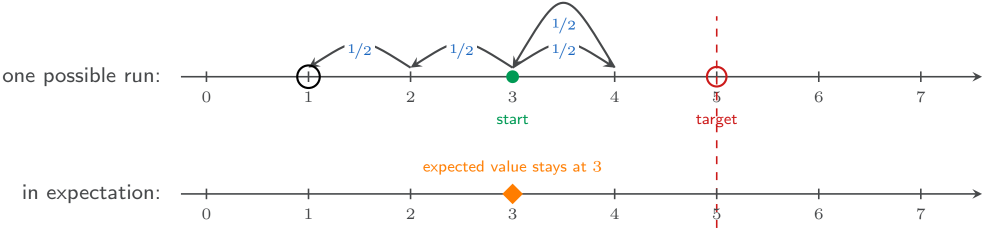

Drift argument:

Drifting Away from the Target

$\mathcal{P}_{\mathsf{rw}}$: $\mathsf{s}(x) \to \{ {}^{1/2}x,\ {}^{1/2}\mathsf{s}(\mathsf{s}(x)) \}$ start: $\mathsf{s}^3(0)$, target: $\mathsf{s}^5(0)$

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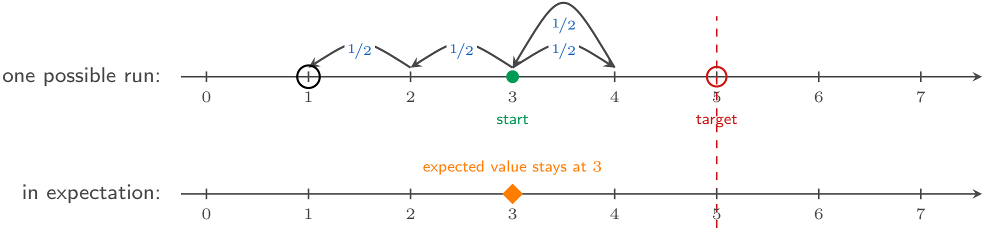

Drift argument: stop every run once it reaches value ≥ 5 (this happens with probability p^*)

Drifting Away from the Target

$\mathcal{P}_{\mathsf{rw}}$: $\mathsf{s}(x) \to \{{}^{1/2}x,\ {}^{1/2}\mathsf{s}(\mathsf{s}(x))\}$ start: $\mathsf{s}^3(0)$, target: $\mathsf{s}^5(0)$

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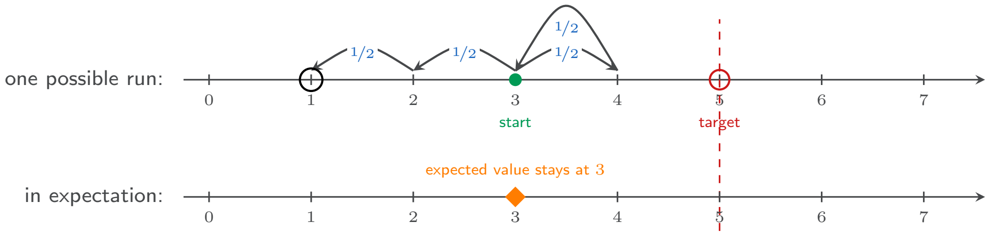

Drift argument: stop every run once it reaches value ≥ 5 (this happens with probability p^*)
"Expected stopped value" $\geq p^* \cdot 5 + (1 - p^*) \cdot 0$, and

Drifting Away from the Target

$\mathcal{P}_{\mathsf{rw}}$: $\mathsf{s}(x) \to \{{}^{1/2}x,\ {}^{1/2}\mathsf{s}(\mathsf{s}(x))\}$ start: $\mathsf{s}^3(0)$, target: $\mathsf{s}^5(0)$

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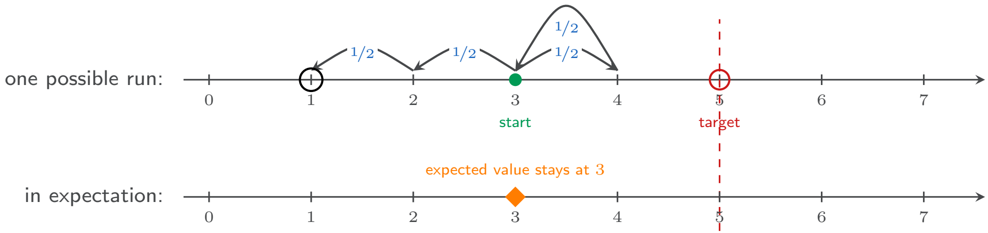

Drift argument: stop every run once it reaches value ≥ 5 (this happens with probability p^*)

"Expected stopped value" $\geq\ p^* \cdot 5 + (1 - p^*) \cdot 0$, and

"Expected stopped value" $\leq\ 3$

Drifting Away from the Target

$\mathcal{P}_{\mathsf{rw}}$: $\mathsf{s}(x) \to \{{}^{1/2}x,\ {}^{1/2}\mathsf{s}(\mathsf{s}(x))\}$ start: $\mathsf{s}^3(0)$, target: $\mathsf{s}^5(0)$

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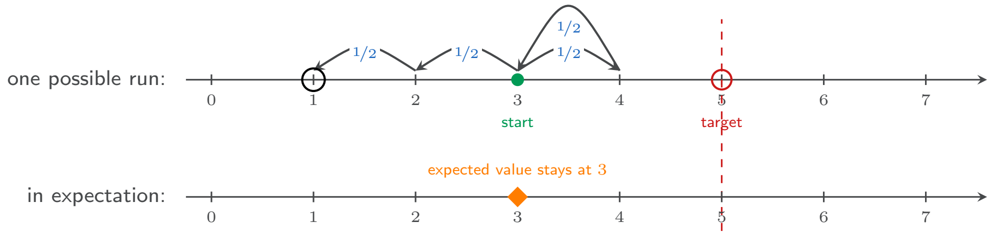


Drift argument: stop every run once it reaches value ≥ 5 (this happens with probability p^*)

"Expected stopped value" $\geq p^* \cdot 5 + (1 - p^*) \cdot 0$, and

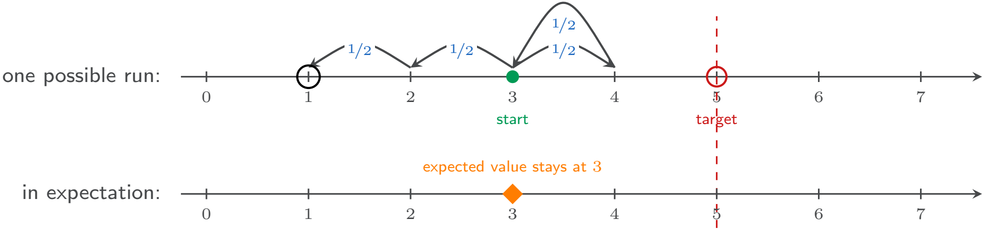
"Expected stopped value" ≤ 3

$\implies \quad p^* \leq {}^{3}/_{5}$

Drifting Away from the Target

$\mathcal{P}_{\mathsf{rw}}$: $\mathsf{s}(x) \to \{{}^{1/2}x,\ {}^{1/2}\mathsf{s}(\mathsf{s}(x))\}$ start: $\mathsf{s}^3(0)$, target: $\mathsf{s}^5(0)$

W places all terms on the number line: start at $W(\mathsf{s}^3(0)) = 3$, target at $W(\mathsf{s}^5(0)) = 5 > 3$



Drift argument: stop every run once it reaches value $\geq W(t)$ (this happens with probability p^*)

"Expected stopped value" $\geq p^* \cdot W(t) + (1 - p^*) \cdot 0$, and

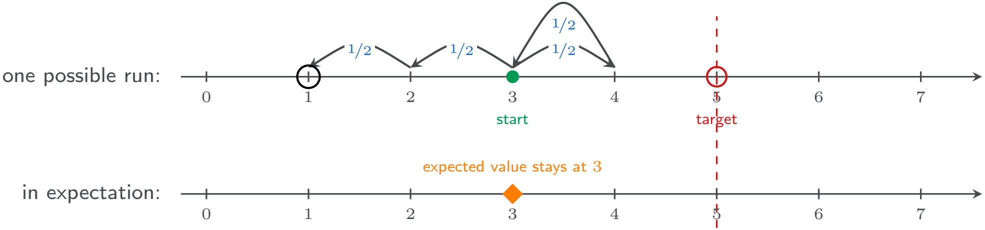
"Expected stopped value" $\leq W(s)$

$$\Longrightarrow \quad p^* \leq {}^{W(s)}/_{W(t)}$$

Drifting Away from the Target

$\mathcal{P}_{\mathsf{rw}}$: $\mathsf{s}(x) \to \{{}^{1/2}x,\ {}^{1/2}\mathsf{s}(\mathsf{s}(x))\}$ start: $\mathsf{s}^3(0)$, target: $\mathsf{s}^5(0)$

W places all terms on the number line: start at $W(\mathsf{s}^3(0)) = 3$, target at $W(\mathsf{s}^5(0)) = 5 > 3$



Upper Bound via Weight Functions

If $W(t) \geq W(s)$ and $W(t) > 0$, then $\mathbb{P}^{\max}(s \to_{\mathcal{P}}^{*} t) \leq \dfrac{W(s)}{W(t)}$

Ranking Functions – Examples

$\mathcal{P}_{\mathsf{rw}}$: $\mathsf{s}(x) \to \{{}^{1/2}x,\ {}^{1/2}\mathsf{s}(\mathsf{s}(x))\}$ weight function $W(\mathsf{0}) = 0$, $W(\mathsf{s}) = 1$

bound $\mathbb{P}^{\max}(s \to_{\mathcal{P}}^{*} t) \leq W(s) \,/\, W(t)$ whenever $W(t) \geq W(s) > 0$ and W supermartingale

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Reach $\mathsf{s}^3(0)$ from $\mathsf{s}(0)$: $W(\mathsf{s}(0)) = 1$, $W(\mathsf{s}^3(0)) = 3$ $\Longrightarrow$ $\mathsf{s}(0) \dashrightarrow^{\mathcal{P}_{\mathsf{rw}}}_{\leq 1/3} \mathsf{s}^3(0)$

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Reach $\mathsf{s}^3(0)$ from $\mathsf{s}(0)$: $W(\mathsf{s}(0)) = 1$, $W(\mathsf{s}^3(0)) = 3$ $\implies$ $\mathsf{s}(0) \dashrightarrow^{\mathcal{P}_{\mathsf{rw}}}_{\leq 1/3} \mathsf{s}^3(0)$

Reach $\mathsf{s}^5(0)$ from $\mathsf{s}(0)$: $W(\mathsf{s}(0)) = 1$, $W(\mathsf{s}^5(0)) = 5$ $\implies$ $\mathsf{s}(0) \dashrightarrow^{\mathcal{P}_{\mathsf{rw}}}_{\leq 1/5} \mathsf{s}^5(0)$

Ranking Functions – Examples

$\mathcal{P}_{\mathsf{rw}}$: $\mathsf{s}(x) \to \{{}^{1/2}x,\ {}^{1/2}\mathsf{s}(\mathsf{s}(x))\}$ weight function $W(0) = 0$, $W(\mathsf{s}) = 1$

bound $\mathbb{P}^{\max}(s \to_{\mathcal{P}}^* t) \leq W(s) \,/\, W(t)$ whenever $W(t) \geq W(s) > 0$ and W supermartingale

Reach $\mathsf{s}^3(0)$ **from** $\mathsf{s}(0)$**:** $W(\mathsf{s}(0)) = 1$, $W(\mathsf{s}^3(0)) = 3$ $\implies \mathsf{s}(0) \dashrightarrow^{\mathcal{P}_{\mathsf{rw}}}_{\leq 1/3} \mathsf{s}^3(0)$

Reach $\mathsf{s}^5(0)$ **from** $\mathsf{s}(0)$**:** $W(\mathsf{s}(0)) = 1$, $W(\mathsf{s}^5(0)) = 5$ $\implies \mathsf{s}(0) \dashrightarrow^{\mathcal{P}_{\mathsf{rw}}}_{\leq 1/5} \mathsf{s}^5(0)$

Reach $\mathsf{s}^5(x)$ **from** $\mathsf{s}^3(x)$**:** $W(\mathsf{s}^3(x)) = x + 3 < x + 5$ $\implies p = \sup_{n \in \mathbb{N}} {}^{(n+3)}/_{(n+5)} = 1$

Ranking Functions – Examples

$\mathcal{P}_{\mathsf{rw}}$: $\mathsf{s}(x) \to \{{}^{1/2}x,\ {}^{1/2}\mathsf{s}(\mathsf{s}(x))\}$ weight function $W(0) = 0,\quad W(\mathsf{s}) = 1$

bound $\mathbb{P}^{\max}(s \to_{\mathcal{P}}^{*} t) \leq W(s) \,/\, W(t)$ whenever $W(t) \geq W(s) > 0$ and W supermartingale

Reach $\mathsf{s}^3(0)$ **from** $\mathsf{s}(0)$**:** $W(\mathsf{s}(0)) = 1,\ W(\mathsf{s}^3(0)) = 3$ $\implies \mathsf{s}(0) \dashrightarrow^{\mathcal{P}_{\mathsf{rw}}}_{\leq 1/3} \mathsf{s}^3(0)$

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each fixed n gives ${}^{n+3}/_{n+5} < 1$, but the bound must hold for *every* instantiation $\rightsquigarrow \sup = 1$

Conclusion

- Lifted **reachability analysis** from TRSs to **probabilistic** TRSs
- Reachability becomes *quantitative*: $\mathbb{P}^{\max}(s \to_{\mathcal{P}}^{*} t)$
- Two techniques to compute **upper bounds**: $s \dashrightarrow_{\leq p}^{\mathcal{P}} t$:

Via MDPs

cut + narrowing $\rightsquigarrow$ finite MDP

tighter with larger cut height n

Via Ranking Functions

weight function + supermartingale

via the Stopping Theorem

Future Work

- Implement both techniques in AProVE and evaluate (use Storm for MDPs)
- Relate the two techniques
- Develop a modular framework for reachability analysis