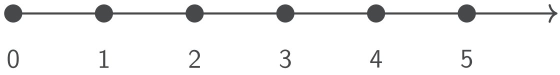
Disproving (Positive) Almost-Sure Termination of Probabilistic Term Rewriting

Jan-Christoph Kassing, Henri Nagel, Alexander Schlecht, and Jürgen Giesl
RWTH Aachen University
27.07.2026

Random Walk



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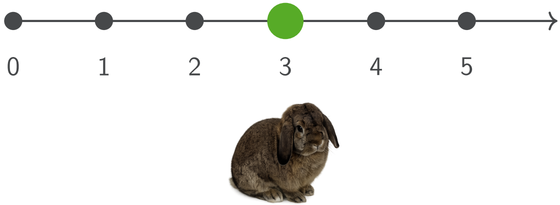

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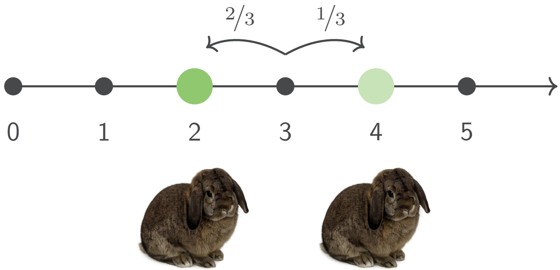

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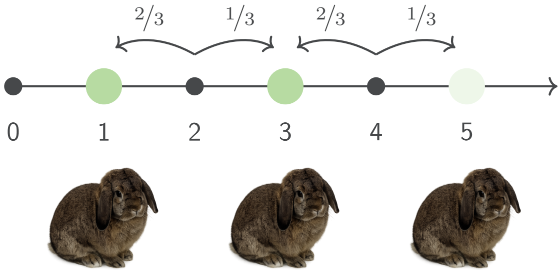

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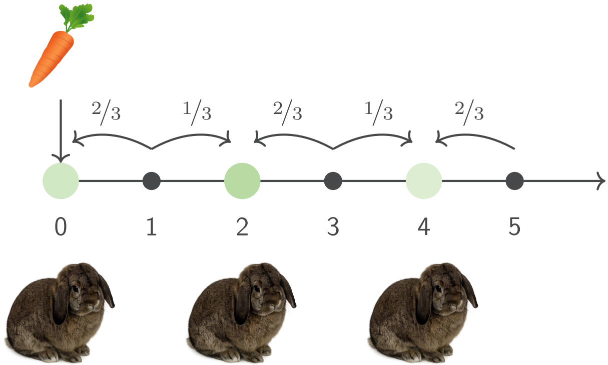
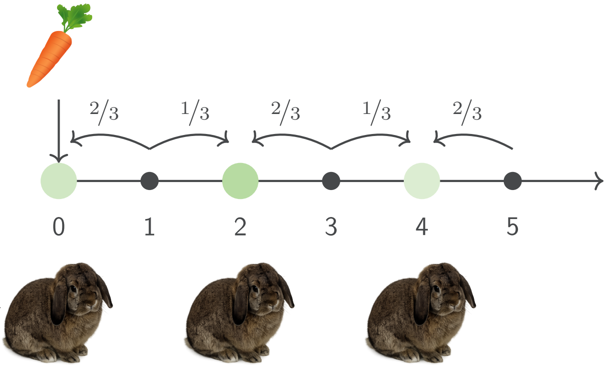


Random Walk

$x \leftarrow 3$
while $x > 0$ **do**
$\quad x \leftarrow x - 1 \oplus_{2/3} x \leftarrow x + 1;$



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$\quad \to \{{}^{2}/_{3} : \mathsf{bunny}(x), {}^{1}/_{3} : \mathsf{bunny}(\mathsf{s}(\mathsf{s}(x)))\}$

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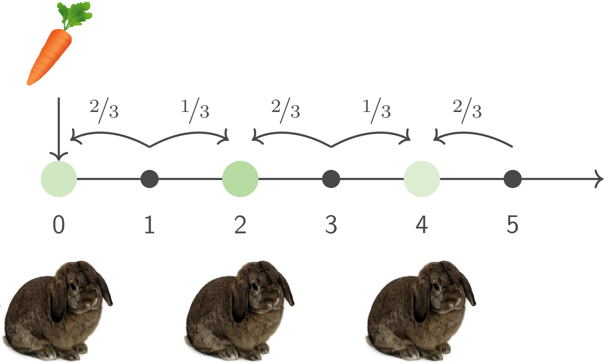

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This Talk!

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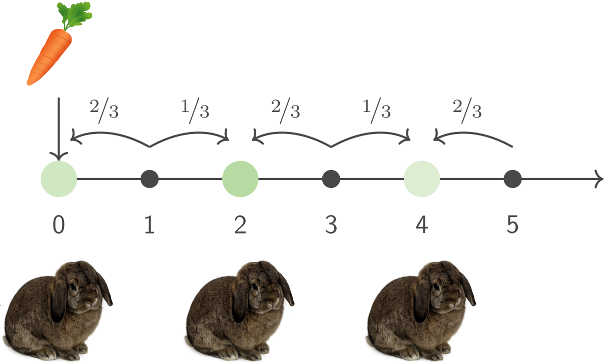

$\mathsf{bunny}(\mathsf{s}(x))$

$\quad \to \{2/3 : \mathsf{bunny}(x), 1/3 : \mathsf{bunny}(\mathsf{s}(\mathsf{s}(x)))\}$

Terms represent arbitrary data structures, not just numbers:

$$[1, 2] \quad = \quad \mathsf{cons}(\mathsf{s}(0), \mathsf{cons}(\mathsf{s}(\mathsf{s}(0)), \mathsf{nil}))$$

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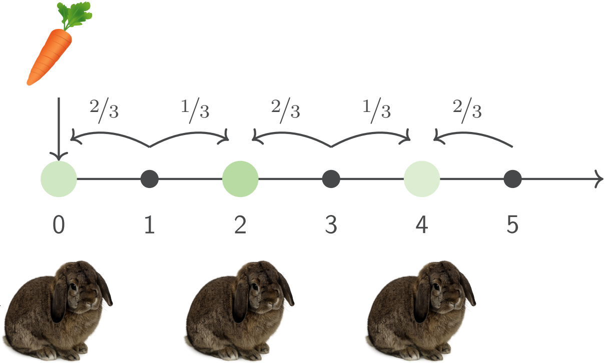

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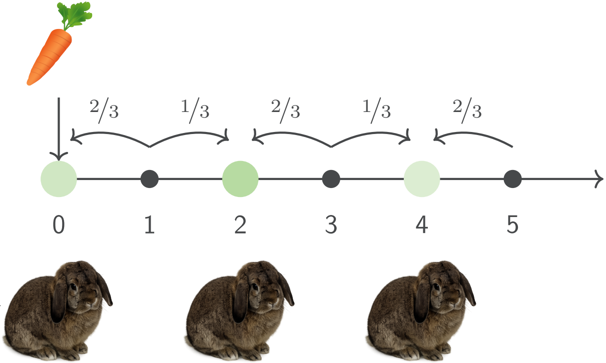

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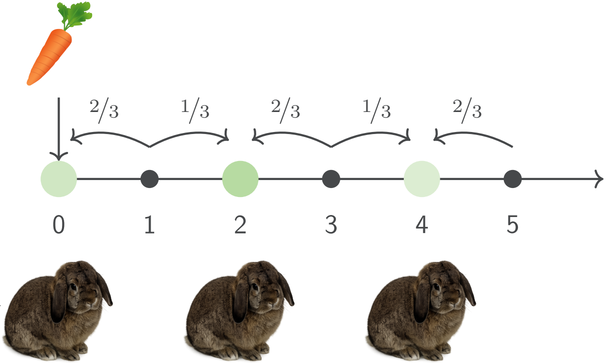

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- Does the bunny (program) always reach the carrot (terminate)?
- What is the probability of reaching the carrot (probability of termination)?
- What is the expected number of steps it takes to reach the carrot (expected runtime)?

Probabilistic Term Rewriting (PTRS)

$\mathcal{P}_{\mathsf{tail}}$: $\qquad \mathsf{tail} \to \left\{\frac{9}{10} : \mathsf{tail},\ \frac{1}{10} : \mathsf{head}\right\}$

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$\mathfrak{T}$: $\boxed{1 \mid \mathsf{tail}}$

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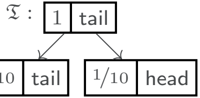

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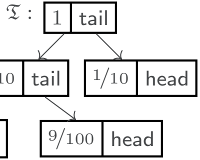

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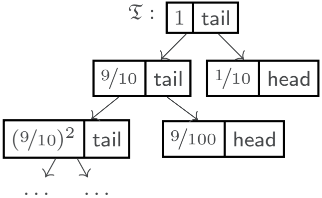

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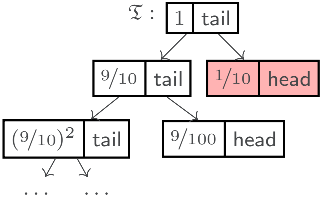
Probability of Termination:

$|\mathfrak{T}| = {}^{1}/_{10}$



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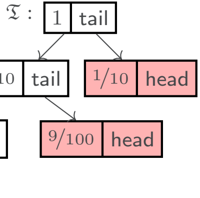
$|\mathfrak{T}| = {}^{1}/_{10} + {}^{9}/_{10} \cdot {}^{1}/_{10}$



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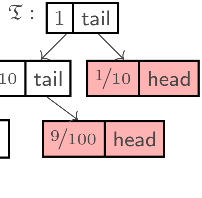

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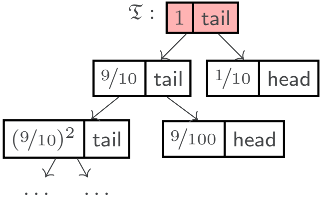

Expected Derivation Height:

$$\mathrm{edh}(\mathfrak{T}) = 1$$

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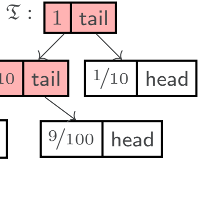
$$\mathrm{edh}(\mathfrak{T}) = 1 + {}^9\!/_{10}$$



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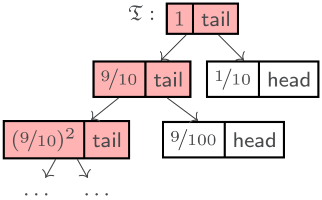

Expected Derivation Height:

$$\mathrm{edh}(\mathfrak{T}) = 1 + {}^{9}/_{10} + ({}^{9}/_{10})^2$$

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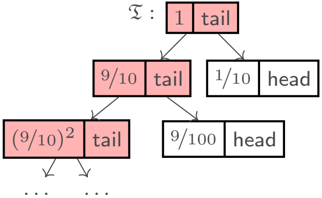
Expected Derivation Height:

$$\mathrm{edh}(\mathfrak{T}) = 1 + {}^9\!/_{10} + ({}^9\!/_{10})^2 + \cdots = \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = 10$$



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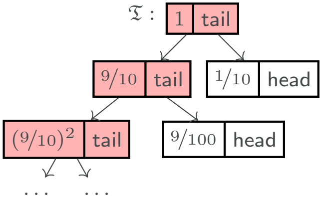

Almost-Sure Termination (AST)

$\mathcal{P}$ is $\mathtt{AST}$ if $|\mathfrak{T}| = 1$ for every $\mathfrak{T}$.

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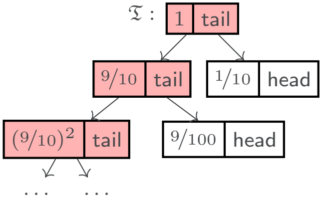

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Positive AST (PAST) ($\mathtt{PAST} \Rightarrow \mathtt{AST}$)

$\mathcal{P}$ is $\mathtt{PAST}$ if $\mathrm{edh}(\mathfrak{T})$ is finite for every $\mathfrak{T}$.

Proving vs. Disproving Termination

Proving AST / PAST of PTRSs ✓

- Ranking Functions
  [Avanzini, Dal Lago, Yamada'20; ...]
- Probabilistic Dependency Pairs
  [Kassing, Spitzer, Giesl'25; ...]

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Disproving AST / PAST of PTRSs ✗ **(this talk)**

- First automated method for PTRSs
- For *imperative* programs:
  - Repulsing Supermartingales
    [Chatterjee, Novotný, Žikelić'17]
  - Fixpoints and Park Induction
    [Takisaka, Oyabu, Urabe, Hasuo'21]

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Deciding AST / PAST of PTRSs ✗

- See VeriProP'26!
- For other formalism:
  - Stochastic Context-Free Grammars / Recursive Markov Chains
    [Etessami, Yannakakis'09; ...]
  - Restricted Probabilistic Higher-Order Recursion Schemes (PHORS)
    [Dal Lago, Fiorillo, Pistone'26; ...]

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Find t, context C, and substitution σ such that

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$\mathcal{P}'_{\mathsf{double}}$:

$$\begin{aligned} \mathsf{double}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{double}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x)))) \end{aligned}$$

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$$
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\downarrow & & \downarrow \\
C[t\sigma] & & \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x)))) \\
\downarrow & & \downarrow \\
C[C[t\sigma]\sigma] & & \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(\mathsf{s}(x)))))) \\
\downarrow & & \downarrow \\
\ldots & & \ldots
\end{array}
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Disproving (P)AST of a PTRS (1.Idea):

Disproving `AST` and `PAST` of a PTRS

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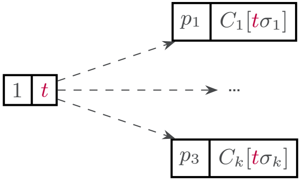

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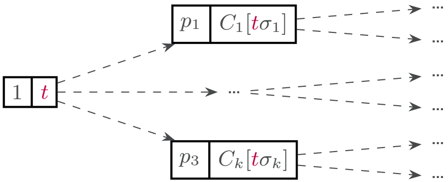

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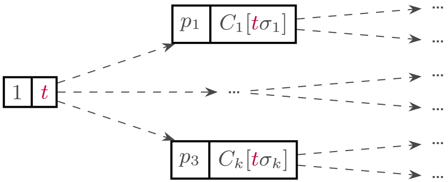

$\mathcal{P}'_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \{1/2 : \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x)))),\ 1/2 : \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(x))))\}$$

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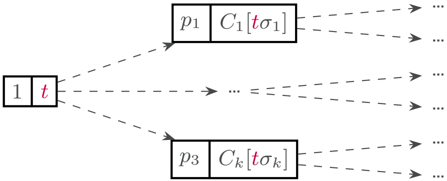

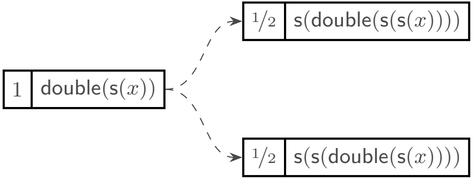

$\mathcal{P}'_{\mathsf{double}}$:

$$\begin{aligned} \mathsf{double}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{double}(\mathsf{s}(x)) &\to \{{}^1\!/\!{}_2 : \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x)))),\ {}^1\!/\!{}_2 : \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(x))))\} \end{aligned}$$

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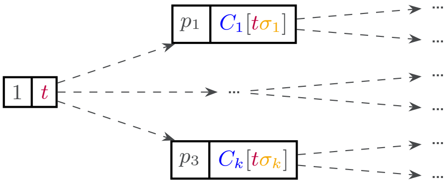

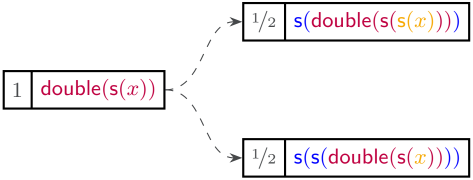

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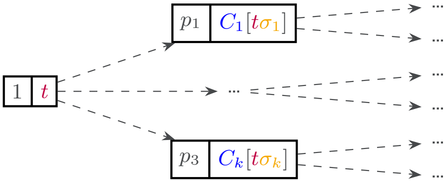
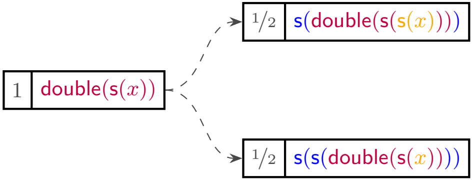




$\mathcal{P}'_{\mathsf{double}}$:

$$\mathsf{double}(\mathsf{0}) \to \mathsf{0}$$

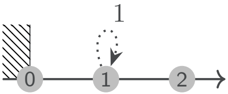
$$\mathsf{double}(\mathsf{s}(x)) \to \{\nicefrac{1}{2} : \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x)))),\ \nicefrac{1}{2} : \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(x))))\}$$

What is a "good" witness for probabilistic systems?

$\rightsquigarrow$ embed suitable random walks

Characterization of Random Walks

Characterization of Random Walks

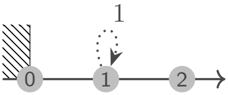


$\mathbb{E} = 0$ (stuck)

Loop Walk μ_1

$$\mu_1(0) = 1$$

Characterization of Random Walks



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$\Longrightarrow$ not AST and not PAST

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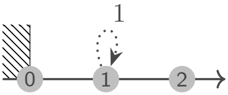

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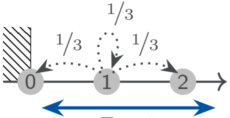


$\mathbb{E} = 0$

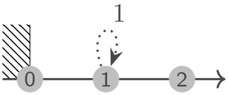
$\mu_2(-1) = {}^1/_3$

$\mu_2(0) = {}^1/_3$

$\mu_2(1) = {}^1/_3$

Symmetric Random Walk μ_2

Characterization of Random Walks



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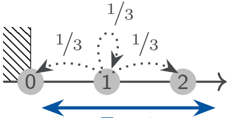

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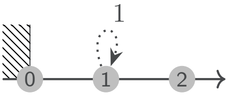

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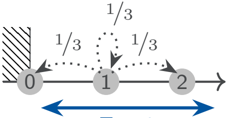


$\mu_2(-1) = 1/3$

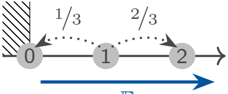
$\mu_2(0) = 1/3$

$\mu_2(1) = 1/3$

$\mathbb{E} = 0$

Symmetric Random Walk μ_2

$\Longrightarrow$ AST and not PAST



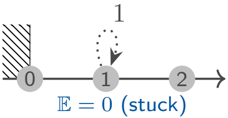
$\mu_3(-1) = 1/3$

$\mu_3(1) = 2/3$

$\mathbb{E}$

Positively Biased Random Walk μ_3

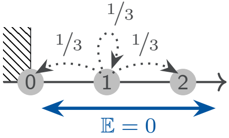
Characterization of Random Walks



$\mu_1(0) = 1$

Loop Walk μ_1

$\Longrightarrow$ not AST and not PAST



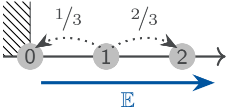
$\mu_2(-1) = {}^1/_3$

$\mu_2(0) = {}^1/_3$

$\mu_2(1) = {}^1/_3$

Symmetric Random Walk μ_2

$\Longrightarrow$ AST and not PAST



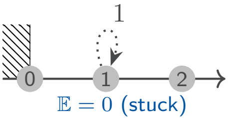
$\mu_3(-1) = {}^1/_3$

$\mu_3(1) = {}^2/_3$

Positively Biased Random Walk μ_3

$\Longrightarrow$ not AST and not PAST

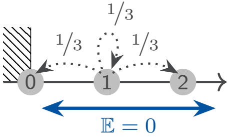
Characterization of Random Walks



$\mu_1(0) = 1$

Loop Walk μ_1

$\Longrightarrow$ not AST and not PAST



$\mu_2(-1) = {}^1/_3$

$\mu_2(0) = {}^1/_3$

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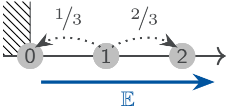
Symmetric Random Walk μ_2

$\Longrightarrow$ AST and not PAST

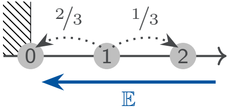


$\mu_3(-1) = {}^1/_3$

$\mu_3(1) = {}^2/_3$

Positively Biased Random Walk μ_3

$\Longrightarrow$ not AST and not PAST

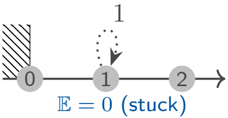


$\mu_4(-1) = {}^2/_3$

$\mu_4(1) = {}^1/_3$

Negatively Biased Random Walk μ_4

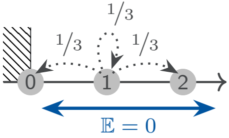
Characterization of Random Walks



$$\mu_1(0) = 1$$

Loop Walk μ_1

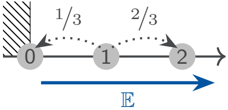
$\implies$ not AST and not PAST



$$\mu_2(-1) = 1/3$$
$$\mu_2(0) = 1/3$$
$$\mu_2(1) = 1/3$$

Symmetric Random Walk μ_2

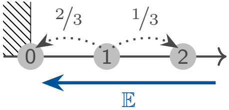
$\implies$ AST and not PAST



$$\mu_3(-1) = 1/3$$
$$\mu_3(1) = 2/3$$

Positively Biased Random Walk μ_3

$\implies$ not AST and not PAST



$$\mu_4(-1) = 2/3$$
$$\mu_4(1) = 1/3$$

Negatively Biased Random Walk μ_4

$\implies$ AST and PAST

Disproving AST and PAST of a PTRS

Disproving (P)AST of a PTRS (2.Idea): Try to embed symmetric or positively biased random walks

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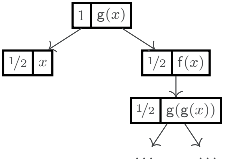
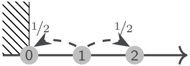
$\mathcal{P}$ Computation



Symmetric Random Walk μ



$\mu(-1) = 1/2$

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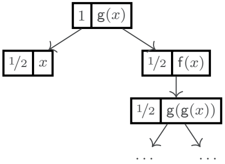
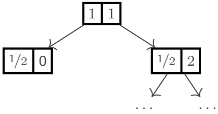
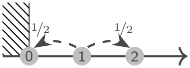


μ Computation



Symmetric Random Walk μ



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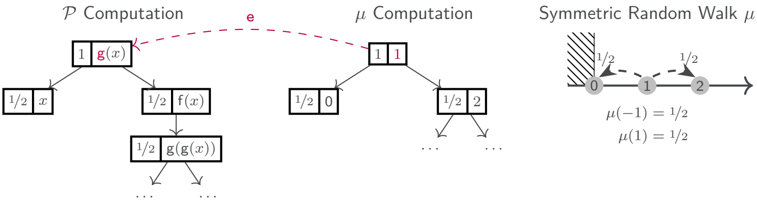

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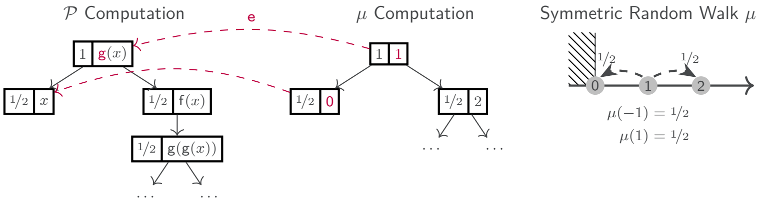

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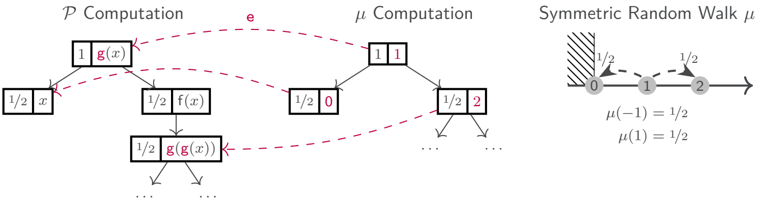

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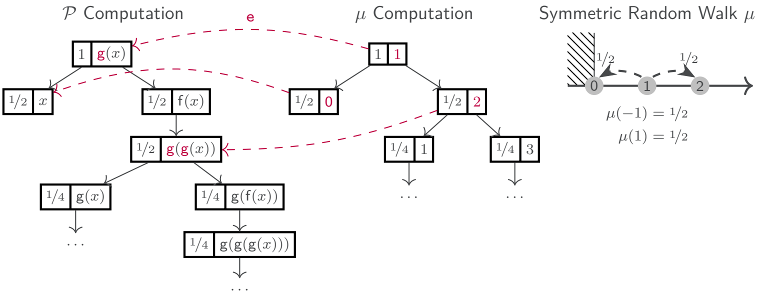

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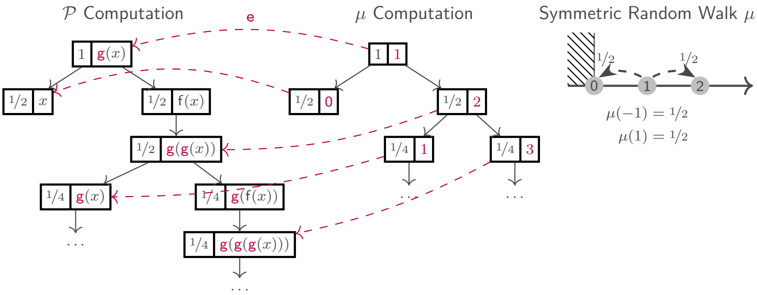

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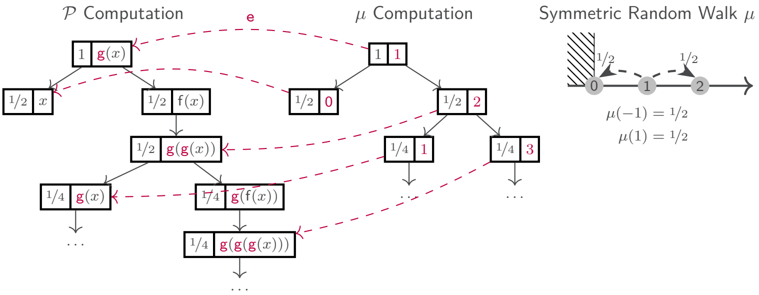

- Embedding e from computation of μ to computation of $\mathcal{P}$

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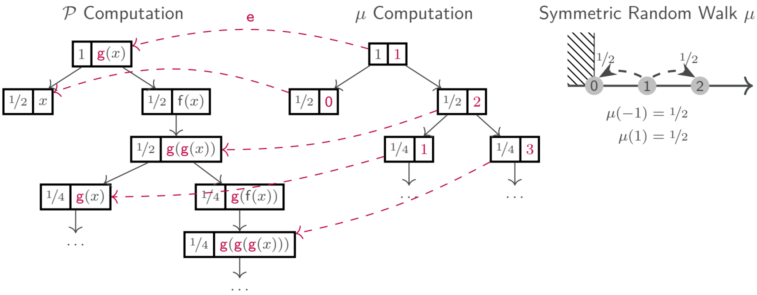

- Embedding e from computation of μ to computation of $\mathcal{P}$
- What and How to Count?

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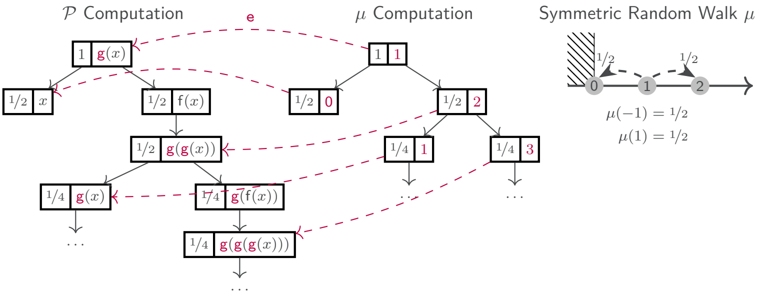

- Embedding e from computation of μ to computation of $\mathcal{P}$
- What and How to Count? $\rightsquigarrow$ Count term occurrences

Counting Term Occurrences

- Find the maximal number t occurs in s (denoted $\mathrm{maxNO}(t, s)$)

Counting Term Occurrences

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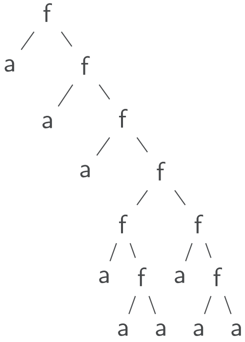
t

s



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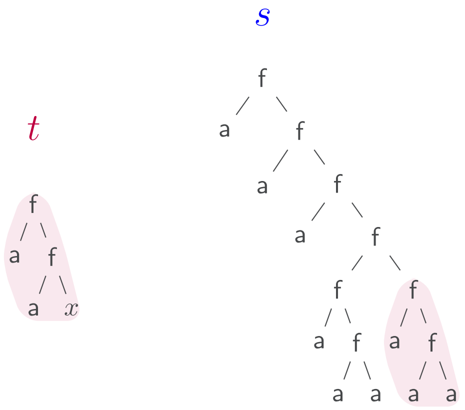

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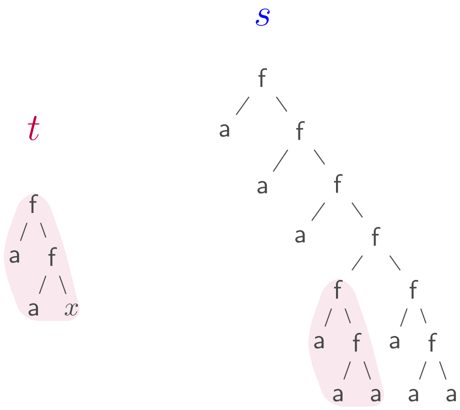

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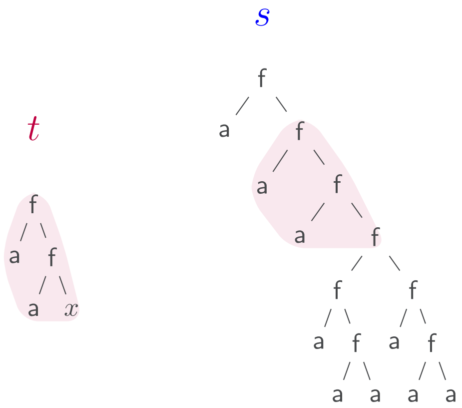

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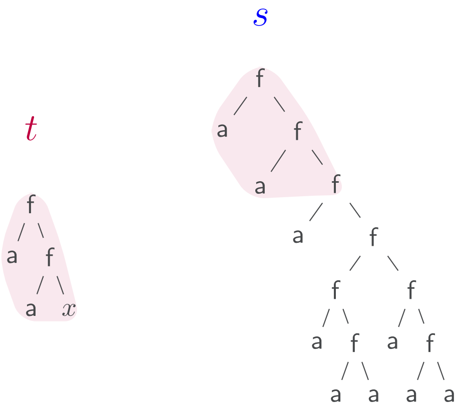
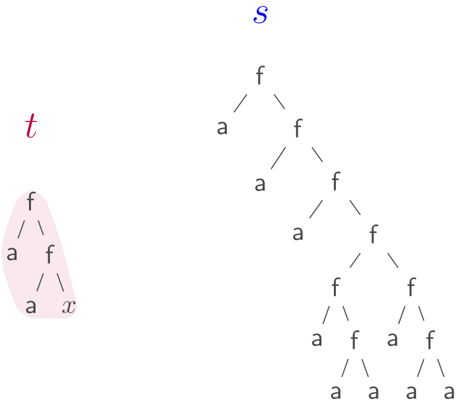


Counting Term Occurrences

- Find the maximal number t occurs in s (denoted $\mathrm{maxNO}(t, s)$)
- Only count non-overlapping occurrences!



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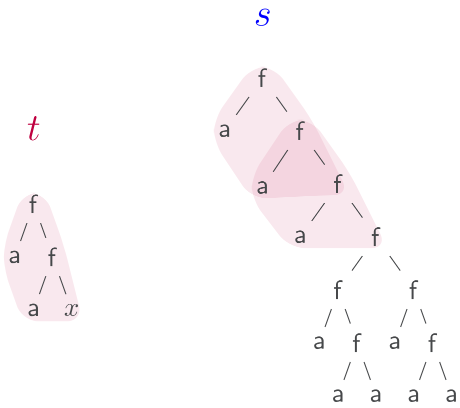

Counting Term Occurrences

- Find the maximal number t occurs in s (denoted $\mathrm{maxNO}(t, s)$)
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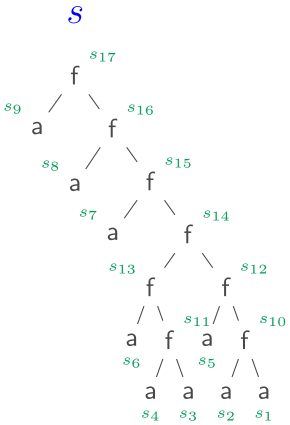
Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$



Counting Term Occurrences

- Find the maximal number t occurs in s (denoted $\mathrm{maxNO}(t, s)$)
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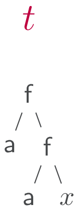
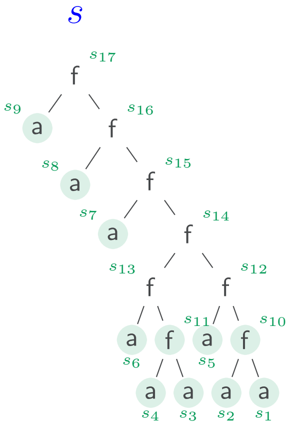
Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$
s_1–s_{11}	0





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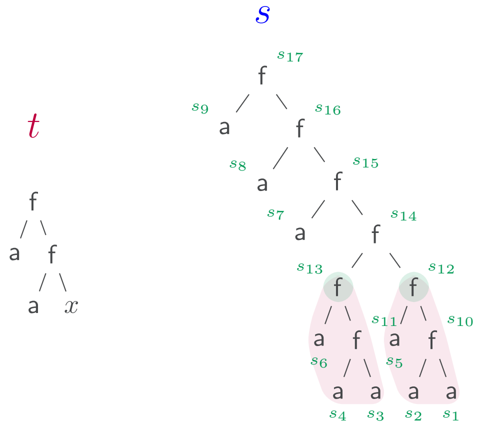
Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$
s_1–s_{11}	0
s_{12}–s_{13}	1



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- Find the maximal number t occurs in s (denoted $\mathrm{maxNO}(t, s)$)
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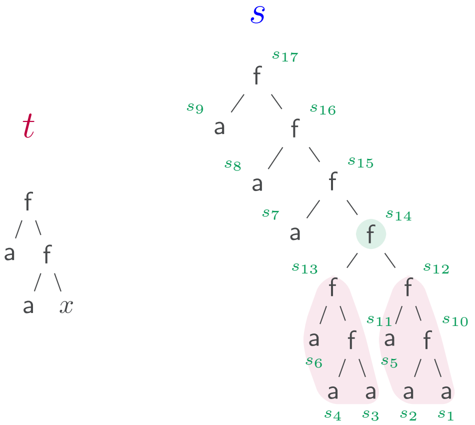
Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$
s_1–s_{11}	0
s_{12}–s_{13}	1
s_{14}	2



Counting Term Occurrences

- Find the maximal number t occurs in s (denoted $\mathrm{maxNO}(t, s)$)
- Only count non-overlapping occurrences!

Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$
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s_{12}–s_{13}	1
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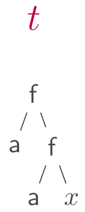

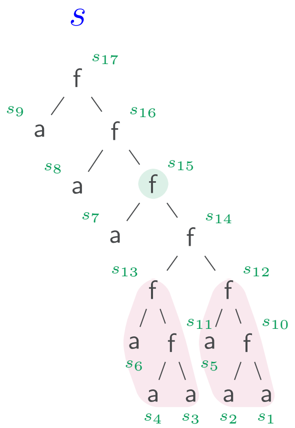

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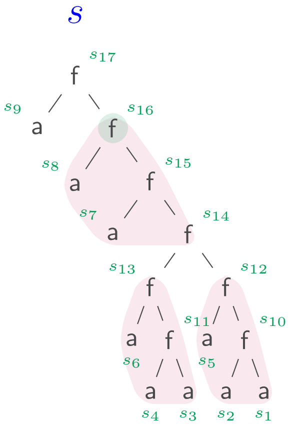

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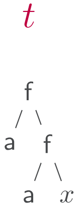

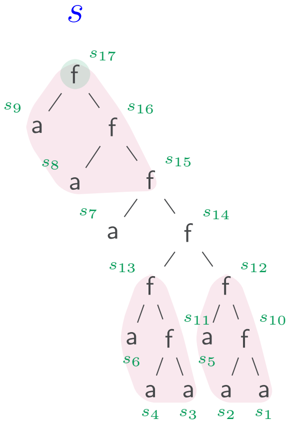

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$$\mathrm{maxNO}(t, s) = \alpha_s = \alpha_{s_{17}} = 3$$

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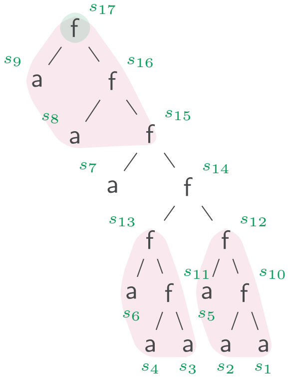
t

s



Construct the Infinite Computation

Theorem: Embedding Random Walks

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$\mathcal{P}$ a PTRS

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Theorem: Embedding Random Walks

$\mathcal{P}$ a PTRS, t a *linear* term

linear $=$ every variable occurs at most once: $\mathsf{g}(\mathsf{s}(x))$ ✓ $\mathsf{f}(x, x)$ ✗

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Theorem: Embedding Random Walks

$\mathcal{P}$ a PTRS, t a *linear* term, and a (finite) computation $\mathfrak{T}$ starting with t.

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Construct the Infinite Computation

Theorem: Embedding Random Walks

$\mathcal{P}$ a PTRS, t a *linear* term, and a (finite) computation $\mathfrak{T}$ starting with t.
If $(\star)$ holds, then $\mathcal{P}$ is not (P)AST.

linear = every variable occurs at most once: $\mathsf{g}(\mathsf{s}(x))$ ✓ $\mathsf{f}(x,x)$ ✗

$$(\star) \qquad \underbrace{\sum_{v \in \mathrm{Leaf}(\mathfrak{T})} p_v \cdot \mathrm{maxNO}(t, t_v)}_{\text{expected number of } t\text{-occurrences after performing } \mathfrak{T}} \quad \overset{}{\underset{(-)}{\geq}} \quad 1$$

How to Find Such Computations and Random Walks?

1. Find loop in $\mathsf{np}(\mathcal{P})$ [Giesl & Thiemann & Schneider-Kamp'05]

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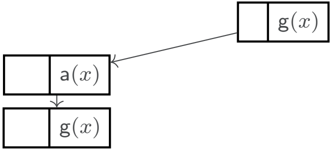
$t = \mathsf{g}(x)$



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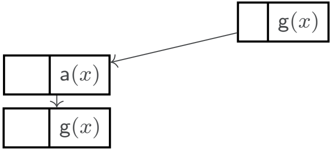

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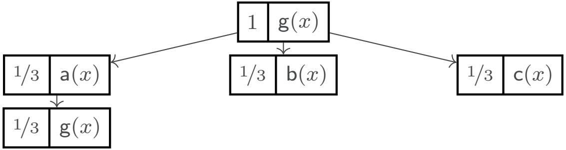

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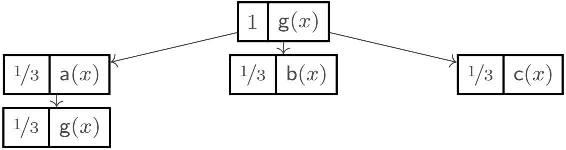

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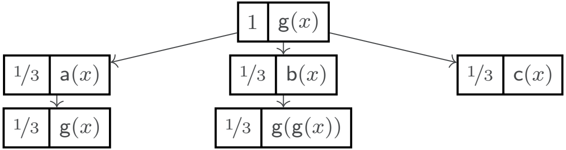

Evaluation

- Implemented in **AProVE**.
- Benchmarks: all 138 PTRSs of the *Termination Competition* $+ 20$ new examples $= 158$.

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Occ. (1) = counting linear, non-overlapping term occurrences, *presented in this talk*
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