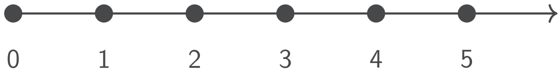
Automatically Analyzing Probabilistic Programs: Proving and Disproving Almost-Sure Termination of Probabilistic Term Rewriting.

Jan-Christoph Kassing
RWTH Aachen University
05.03.2026

Random Walk



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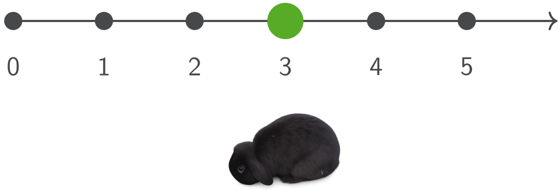

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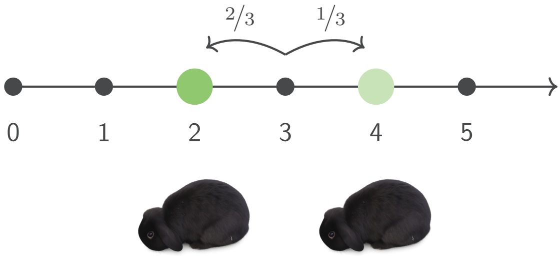

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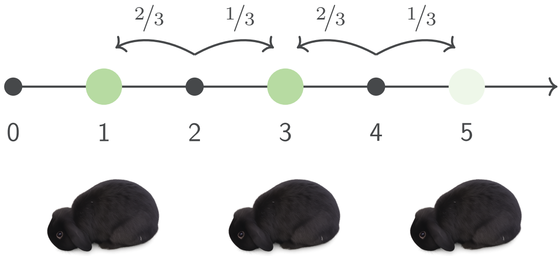

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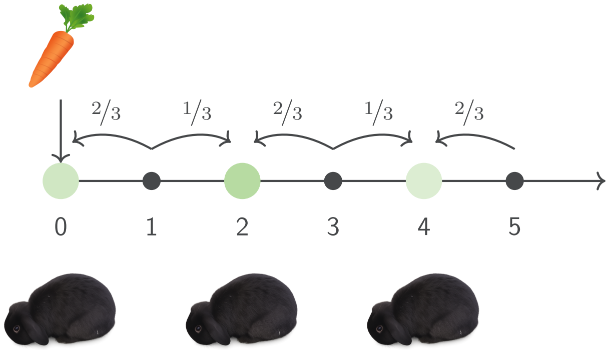

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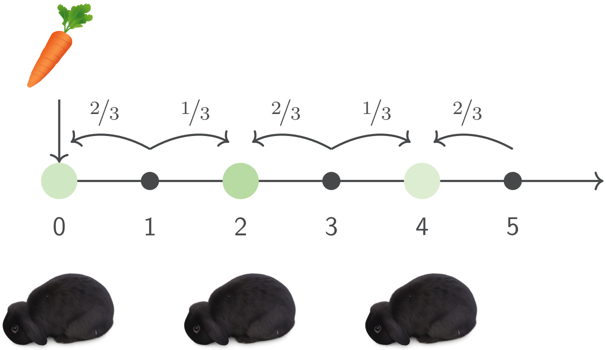
$\quad x \leftarrow x - 1 \oplus_{2/3} x \leftarrow x + 1;$



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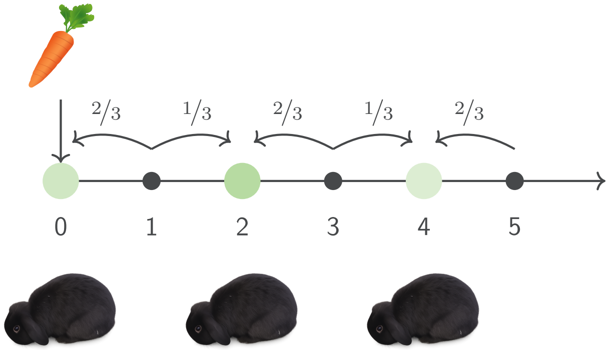

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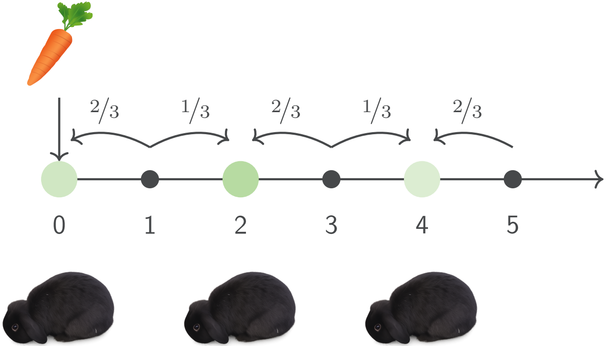

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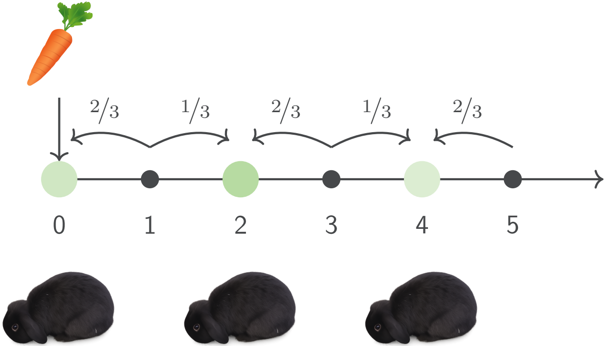

- Does the bunny (program) always reach the carrot (terminate)?
- What is the probability of reaching the carrot (probability of termination)?
- What is the expected number of steps it takes to reach the carrot (expected runtime)?

Two Important Questions

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Monte Carlo Algorithms

Las Vegas Algorithms

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Polynomial Zero Testing

Given: $p(x_1, \ldots, x_n) \in \mathbb{F}_q[x_1, \ldots, x_n]$

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  This is correct with probability 1.

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- If $p(a_1, \ldots, a_n) = 0$, then return *True*.

  This is correct with probability $\geq \frac{d}{q}$, where d is the degree of $p(x_1, \ldots, x_n)$.

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Given: $x_1, \ldots, x_n \in \mathbb{N}$

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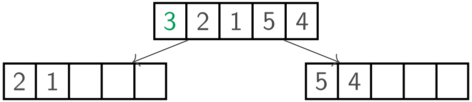

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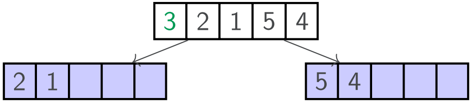

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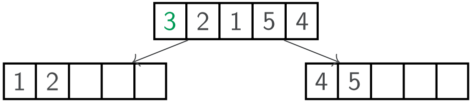

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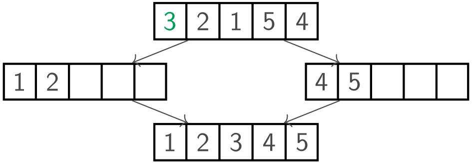

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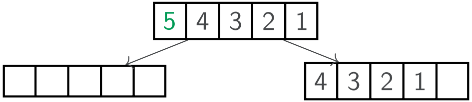

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Prolog

C

Haskell

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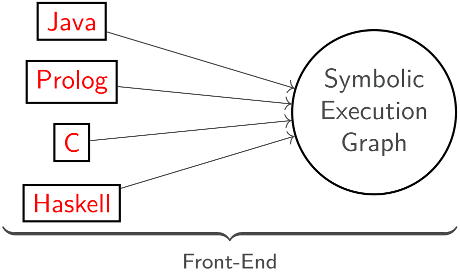

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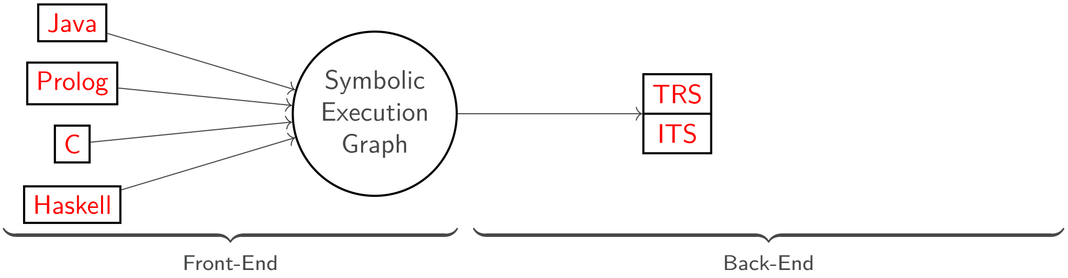

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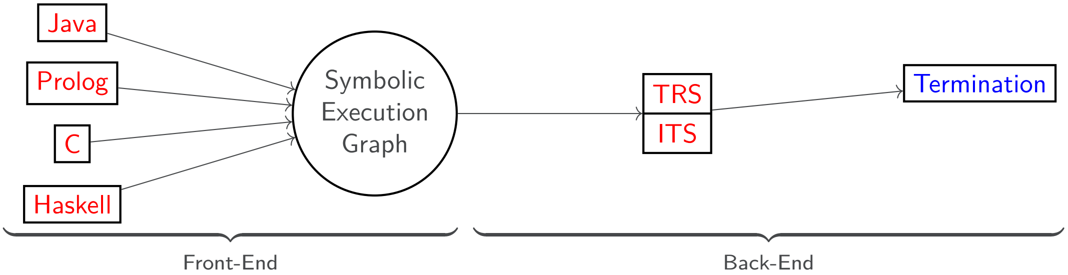

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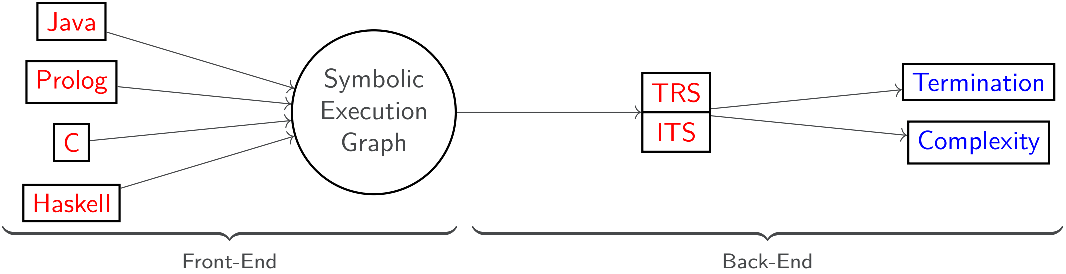

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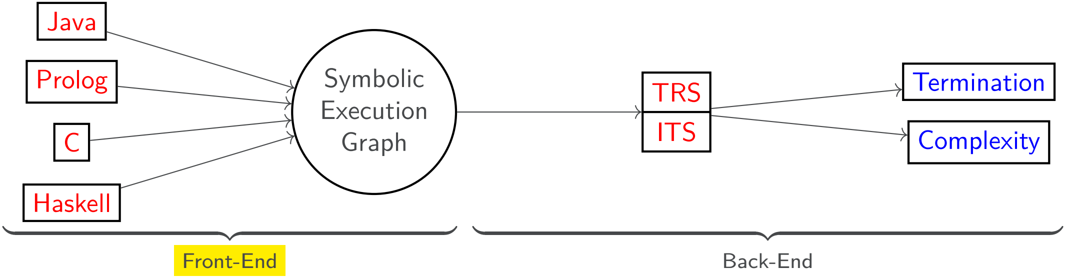
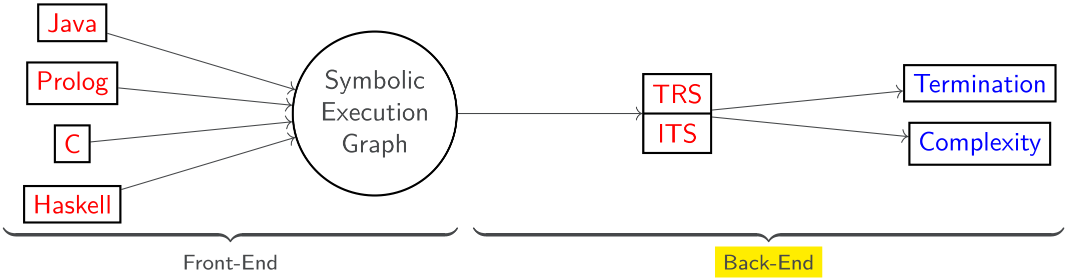


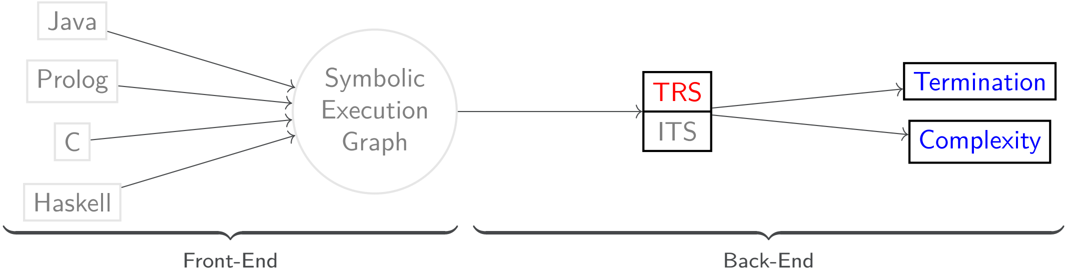
- language-specific features when generating symbolic execution graph

Termination and Complexity Analysis for Programs

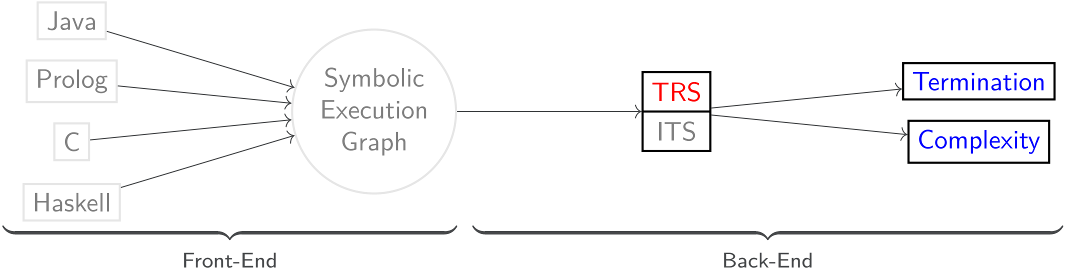


- language-specific features when generating symbolic execution graph
- back-end analyzes Term Rewrite Systems and/or Integer Transition Systems

Termination and Complexity Analysis for Programs

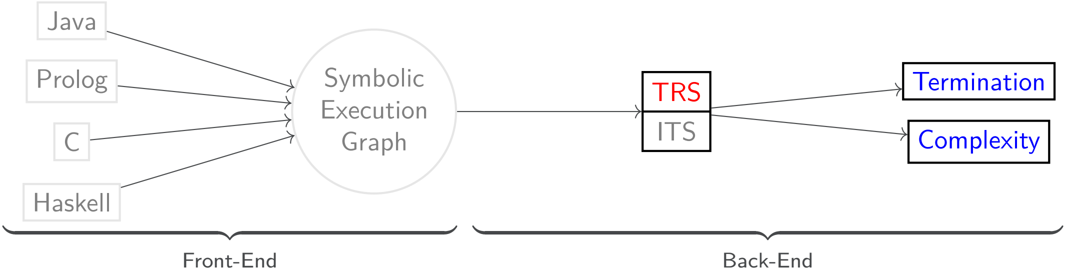


Termination and Complexity Analysis for Programs



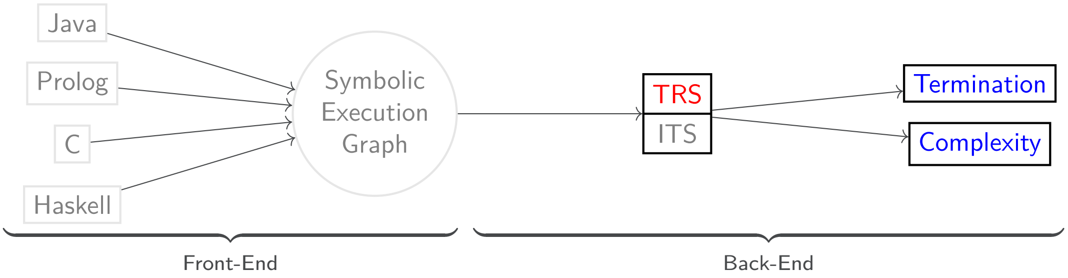
- Proving Termination and Complexity of TRSs

Termination and Complexity Analysis for Programs



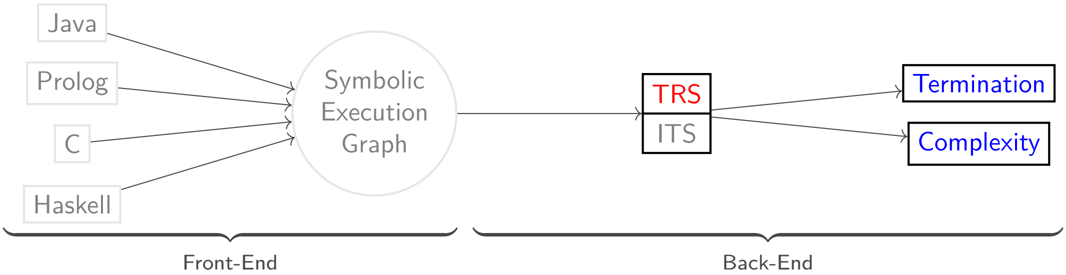
- Proving Termination and Complexity of TRSs
- Proving Termination and Complexity of Probabilistic TRSs

Termination and Complexity Analysis for Programs



- Proving Termination and Complexity of TRSs
- Proving Termination and Complexity of Probabilistic TRSs
- Disproving Termination of Probabilistic TRSs

Termination and Complexity Analysis for Programs



- Proving Termination and Complexity of TRSs
- Proving Termination and Complexity of Probabilistic TRSs
- Disproving Termination of Probabilistic TRSs

Complexity of TRSs

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))$$

Complexity of TRSs

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))$$

$\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))$

Complexity of TRSs

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))$$

$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0))))$$

Complexity of TRSs

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$

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$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0))$$

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Complexity of TRSs

$\mathcal{R}_{\mathsf{double}}$:

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Termination

TRS $\mathcal{R}$ is *terminating* if there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \dots$

$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0)) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(0)$$

Complexity of TRSs

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Runtime Complexity, $\mathrm{rc}_{\mathcal{R}}$

$$\mathrm{rc}_{\mathcal{R}}(n) = \sup\{\mathrm{dh}_{\mathcal{R}}(t) \mid \qquad |t| \leq n\}$$

$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0)) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(0)$$

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derivation **h**eight

$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0)) \to_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(0)$$

$\mathrm{dh}_{\mathcal{R}}(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) =$ "max number of steps" $= 3$

Complexity of TRSs

$\mathcal{R}_{\mathsf{double}}$:

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Innermost Runtime Complexity, $\mathrm{rc}_{\mathcal{R}}$

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$\mathrm{dh}_{\mathcal{R}}\,(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) =$ "max number of steps" $= 3$

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$\mathsf{double}(\mathsf{double}(\mathsf{s}(0)))$

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$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0)) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(0)$$

$\mathrm{dh}_{\mathcal{R}}\,(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) = \text{“max number of steps”} = 3$

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$\mathrm{dh}_{\mathcal{R}}\,(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) = \text{“max number of steps”} = 3$

$$\mathsf{double}(\mathsf{double}(\mathsf{s}(0))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{double}(\mathsf{s}(\mathsf{double}(0))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \ldots$$

Complexity of TRSs

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$\mathrm{dh}_{\mathcal{R}}\,(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) = \text{“max number of steps”} = 3$

Basic Terms, $\mathcal{T}_B$

Complexity of TRSs

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))$$

Innermost Runtime Complexity, $\mathrm{rc}_{\mathcal{R}}$

$$\mathrm{rc}_{\mathcal{R}}(n) = \sup\{\mathrm{dh}_{\mathcal{R}}(t) \mid \qquad |t| \leq n\}$$

Defined Symbols Σ_D: double,

$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0)) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(0)$$

$\mathrm{dh}_{\mathcal{R}}(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) =$ "max number of steps" $= 3$

Basic Terms, $\mathcal{T}_B$

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Defined Symbols Σ_D: double, Constructors Σ_C: s, 0

$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \xrightarrow{\mathrm{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \xrightarrow{\mathrm{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0)) \xrightarrow{\mathrm{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(0)$$

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Innermost Runtime Complexity, $\mathrm{rc}_{\mathcal{R}}$

$$\mathrm{rc}_{\mathcal{R}}(n) = \sup\{\mathrm{dh}_{\mathcal{R}}(t) \mid t \in \mathcal{T}_B, |t| \leq n\}$$

Defined Symbols Σ_D: double, Constructors Σ_C: s, 0

$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0)) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(0)$$

$\mathrm{dh}_{\mathcal{R}}(\mathsf{double}(\mathsf{s}(\mathsf{s}(0))))$ = "max number of steps" $= 3$

Basic Terms, $\mathcal{T}_B$

Terms $f(c_1, \ldots, c_n) \in \mathcal{T}_B$ are *basic* if $f \in \Sigma_D$ and all c_i are constructor terms.

Complexity of TRSs

$\mathcal{R}_{\mathsf{double}}$:

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$$\mathrm{rc}_{\mathcal{R}}(n) = \sup\{\mathrm{dh}_{\mathcal{R}}(t) \mid t \in \mathcal{T}_B, |t| \leq n\}$$

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$\mathrm{dh}_{\mathcal{R}}(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) =$ "max number of steps" $= 3$ $\quad \mathrm{rc}_{\mathcal{R}_{\mathsf{double}}}(n) = n - 1$

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Complexity of TRSs

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$$\mathrm{rc}_{\mathcal{R}}(n) = \sup\{\mathrm{dh}_{\mathcal{R}}(t) \mid t \in \mathcal{T}_B, |t| \leq n\}$$

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$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(\mathsf{double}(0)) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \mathsf{s}^4(0)$$

$\mathrm{dh}_{\mathcal{R}}\left(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))\right) =$ "max number of steps" $= 3$ $\quad \mathrm{rc}_{\mathcal{R}_{\mathsf{double}}}(n) = n - 1 \in \mathcal{O}(n^1)$

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Terms $f(c_1, \ldots, c_n) \in \mathcal{T}_B$ are *basic* if $f \in \Sigma_D$ and all c_i are constructor terms.

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Innermost Runtime Complexity, $\mathrm{rc}_{\mathcal{R}}$

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Defined Symbols Σ_D: double, Constructors Σ_C: s, 0

$$\mathsf{double}(\mathsf{s}(\mathsf{s}(0))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\text{double}}} \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(0)))) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\text{double}}} \mathsf{s}^4(\mathsf{double}(0)) \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\text{double}}} \mathsf{s}^4(0)$$

$\mathrm{dh}_{\mathcal{R}}(\mathsf{double}(\mathsf{s}(\mathsf{s}(0)))) =$ "max number of steps" $= 3$ $\quad \mathrm{rc}_{\mathcal{R}_{\text{double}}}(n) = n - 1 \in \mathcal{O}(n^1)$ $\quad \mathrm{rc}_{\mathcal{R}_{\text{double}}} = \mathrm{Pol}_1$

Basic Terms, $\mathcal{T}_B$

Terms $f(c_1, \ldots, c_n) \in \mathcal{T}_B$ are *basic* if $f \in \Sigma_D$ and all c_i are constructor terms.

Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))$$

Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

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Natural & Monotonic Polynomial Interpretation $\mathcal{I}$

- natural: $\mathcal{I}_f(x_1, \ldots, x_n)$ is a polynomial with natural coefficients for every function symbol $f \in \Sigma$
- monotonic: $x > y$ implies $\mathcal{I}_f(\ldots, x, \ldots) > \mathcal{I}_f(\ldots, y, \ldots)$ for every function symbol $f \in \Sigma$

Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))$$

$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

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Proving Termination With Natural & Monotonic $\mathcal{I}$ [Lankford'79]

$\mathcal{R}$ is terminating if for all rules $\ell \to r : \mathcal{I}(\ell) > \mathcal{I}(r)$

Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

$$\mathcal{I}(\mathsf{double}(0)) > \mathcal{I}(0)$$
$$\mathcal{I}(\mathsf{double}(\mathsf{s}(x))) > \mathcal{I}(\mathsf{s}(\mathsf{s}(\mathsf{double}(x))))$$

$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

Proving Termination With Natural & Monotonic $\mathcal{I}$ [Lankford'79]

$\mathcal{R}$ is terminating if for all rules $\ell \to r : \mathcal{I}(\ell) > \mathcal{I}(r)$

Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

$$2 \cdot \mathcal{I}(0) + 1 > 1$$
$$2 \cdot \mathcal{I}(\mathsf{s}(x)) + 1 > \mathcal{I}(\mathsf{s}(\mathsf{double}(x))) + 1$$

$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

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$\mathcal{R}$ is terminating if for all rules $\ell \to r : \mathcal{I}(\ell) > \mathcal{I}(r)$

Proving Termination

$\mathcal{R}_{\text{double}}$:

$$2 \cdot 1 + 1 > 1$$
$$2 \cdot (x + 1) + 1 > 2 \cdot x + 2$$

$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\text{double}}(x) = 2x + 1$$

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Proving Termination

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$\mathcal{R}$ is terminating if for all rules $\ell \to r : \mathcal{I}(\ell) > \mathcal{I}(r)$

$$t_0 \quad \overset{\mathsf{i}}{\to}_{\mathcal{R}_{\mathsf{double}}} \quad t_1 \quad \overset{\mathsf{i}}{\to}_{\mathcal{R}_{\mathsf{double}}} \quad t_2 \quad \overset{\mathsf{i}}{\to}_{\mathcal{R}_{\mathsf{double}}} \quad \dots$$

Proving Termination

$\mathcal{R}_{\mathsf{double}}$:

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$\mathcal{R}$ is terminating if for all rules $\ell \to r : \mathcal{I}(\ell) > \mathcal{I}(r)$

$$t_0 \quad \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \quad t_1 \quad \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \quad t_2 \quad \xrightarrow{\mathsf{i}}_{\mathcal{R}_{\mathsf{double}}} \quad \ldots$$

$$\mathcal{I}(t_0) \quad > \quad \mathcal{I}(t_1) \quad > \quad \mathcal{I}(t_2) \quad > \quad \ldots$$

Proving Complexity

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(\mathsf{0}) \to 0$$
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$$\mathcal{I}_{\mathsf{0}} = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

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Goal: Infer complexity from the highest degree of $\mathcal{I}$ [Hofbauer & Lautemann'89]

Proving Complexity

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Goal: Infer complexity from the highest degree of $\mathcal{I}$ [Hofbauer & Lautemann'89]

For basic term $t = \mathsf{double}(\mathsf{s}^n(0))$:

Proving Complexity

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(\mathsf{0}) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{double}(x))$$

$$\mathcal{I}_{\mathsf{0}} = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

Goal: Infer complexity from the highest degree of $\mathcal{I}$ [Hofbauer & Lautemann'89]

For basic term $t = \mathsf{double}(\mathsf{s}^n(\mathsf{0}))$: $\qquad \mathcal{I}(t) = 2 \cdot (n+1) + 1$

Proving Complexity

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{double}(x))$$

$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1 \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

Goal: Infer complexity from the highest degree of $\mathcal{I}$ [Hofbauer & Lautemann'89]

For basic term $t = \mathsf{double}(\mathsf{s}^n(0))$: $\mathcal{I}(t) = 2 \cdot (n+1) + 1$

$\rightsquigarrow$ at most linear runtime complexity

Proving Complexity

$\mathcal{R}_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{double}(x))$$

$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\mathsf{s}}(x) = 2x \qquad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$$

Goal: Infer complexity from the highest degree of $\mathcal{I}$ [Hofbauer & Lautemann'89]

For basic term $t = \mathsf{double}(\mathsf{s}^n(0))$: $\mathcal{I}(t) = 2 \cdot (\; 2^n \;) + 1$

Proving Complexity

$\mathcal{R}_{\text{double}}$:

$$\text{double}(0) \to 0$$
$$\text{double}(\text{s}(x)) \to \text{s}(\text{double}(x))$$

$$\mathcal{I}_0 = 1 \qquad \mathcal{I}_{\text{s}}(x) = 2x \qquad \mathcal{I}_{\text{double}}(x) = 2x + 1$$

Goal: Infer complexity from the highest degree of $\mathcal{I}$ [Hofbauer & Lautemann'89]

For basic term $t = \text{double}(\text{s}^n(0))$: $\qquad \mathcal{I}(t) = 2 \cdot (\; 2^n \;) + 1$

Complexity Polynomial Interpretation (CPI)

$\mathcal{I}_f(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n + b$ for every constructor $f \in \Sigma_C$ with $b \in \mathbb{N}, a_i \in \{0, 1\}$

Proving Complexity

$\mathcal{R}_{\text{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{double}(x))$$

CPI: $\mathcal{I}_{0} = 1$ $\quad \mathcal{I}_{\mathsf{s}}(x) = x + 1$ $\quad \mathcal{I}_{\mathsf{double}}(x) = 2x + 1$

Goal: Infer complexity from the highest degree of $\mathcal{I}$ [Hofbauer & Lautemann'89]

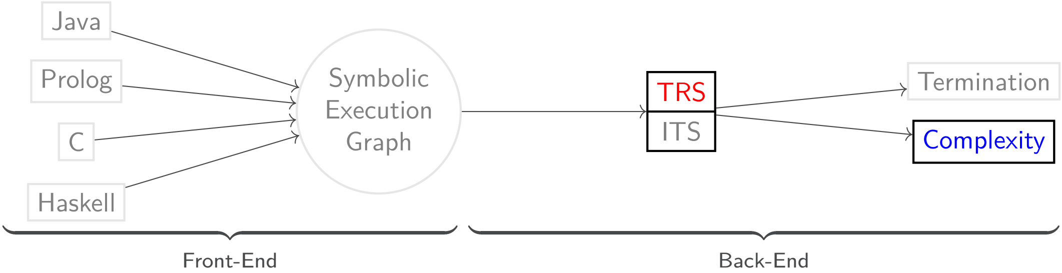
For basic term $t = \mathsf{double}(\mathsf{s}^n(0))$: $\quad \mathcal{I}(t) = 2 \cdot (n+1) + 1$

$\rightsquigarrow$ at most linear runtime complexity

Complexity Polynomial Interpretation (CPI)

$\mathcal{I}_f(x_1, \ldots, x_n) = a_1 x_1 + \ldots + a_n x_n + b$ for every constructor $f \in \Sigma_C$ with $b \in \mathbb{N}, a_i \in \{0, 1\}$

Termination and Complexity Analysis for Programs



- Proving Termination and Complexity of TRSs
- Proving Termination and Complexity of Probabilistic TRSs
- Disproving Termination of Probabilistic TRSs

Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

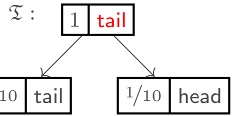
Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

$\mathfrak{T}$: | 1 | tail |

Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \}$

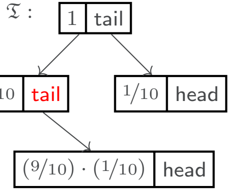
Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$



Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

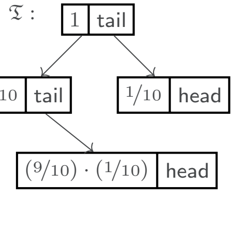
Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$



Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$



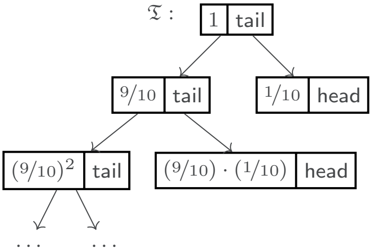
Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Probability of Termination:

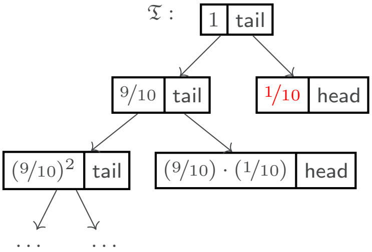
$|\mathfrak{T}|$



Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$



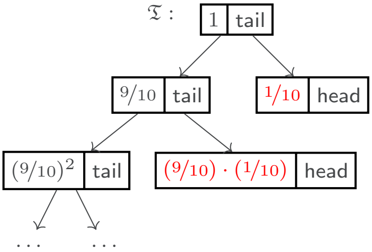
Probability of Termination:

$$|\mathfrak{T}| = {}^{1}\!/_{10} +$$

Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$



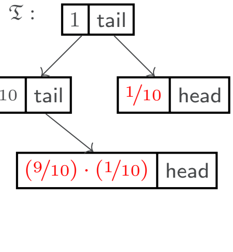
Probability of Termination:

$$|\mathfrak{T}| = {}^{1}/_{10} + {}^{9}/_{10} \cdot {}^{1}/_{10} +$$

Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$



Probability of Termination:

$$|\mathfrak{T}| = {}^1\!/_{10} + {}^9\!/_{10} \cdot {}^1\!/_{10} + ({}^9\!/_{10})^2 \cdot {}^1\!/_{10} + \ldots$$

Almost-Sure Termination Probabilistic TRSs

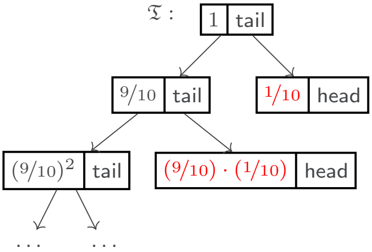
$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Probability of Termination:

$$|\mathfrak{T}| = {}^1\!/_{10} + {}^9\!/_{10} \cdot {}^1\!/_{10} + ({}^9\!/_{10})^2 \cdot {}^1\!/_{10} + \ldots$$

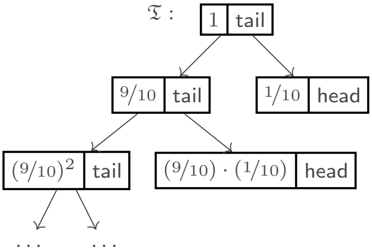
$$= {}^1\!/_{10} \cdot \sum_{n=0}^{\infty} (\tfrac{9}{10})^n = {}^1\!/_{10} \cdot 10 = 1$$



Almost-Sure Termination Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$



Probability of Termination:

$$|\mathfrak{T}| = {}^1/_{10} + {}^9/_{10} \cdot {}^1/_{10} + ({}^9/_{10})^2 \cdot {}^1/_{10} + \ldots$$
$$= {}^1/_{10} \cdot \sum_{n=0}^{\infty} (\tfrac{9}{10})^n = {}^1/_{10} \cdot 10 = 1$$

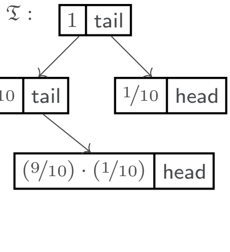
Almost-Sure Termination ($\mathtt{AST}$) [Avanzini & Dal Lago & Yamada'20]

PTRS $\mathcal{P}$ is $\mathtt{AST}$ if $|\mathfrak{T}| = 1$ for every $\mathfrak{T}$.

Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$



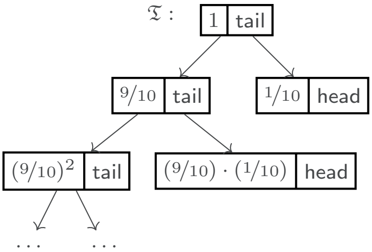
Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

Distribution: $\mu = \{p_1 : t_1, \dots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$\mathrm{edl}(\mathfrak{T})$



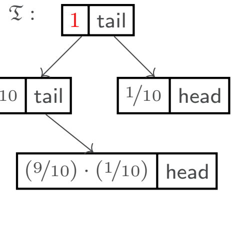
Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 +$$



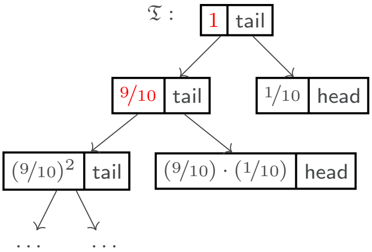
Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} +$$



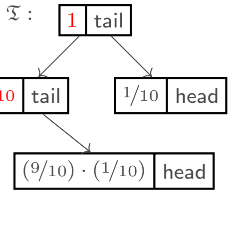
Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: tail $\to \{ \frac{9}{10}$: tail, $\frac{1}{10}$: head$\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \ldots$$



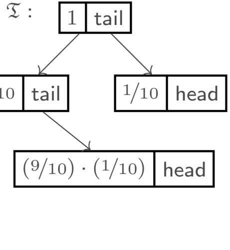
Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\quad \mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

Distribution: $\mu = \{p_1 : t_1, \dots, p_k : t_k\}$ with $p_1 + \dots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + (\frac{9}{10})^2 + \dots = \sum_{n=0}^{\infty} (\frac{9}{10})^n = 10$$



Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: tail $\to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

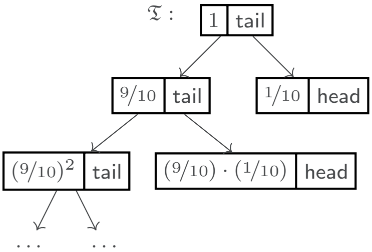
Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \ldots = \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = 10$$

Expected Derivation Height:

$$\mathrm{edh}_{\mathcal{P}_{\mathsf{tail}}}(\mathsf{tail}) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with tail}\}$$



Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

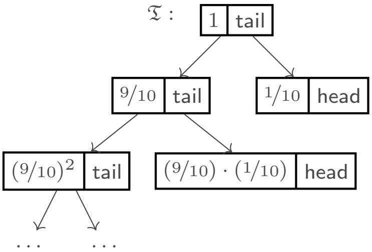
Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \ldots = \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = 10$$

Expected Derivation Height:

$$\mathrm{edh}_{\mathcal{P}_{\mathsf{tail}}}(\mathsf{tail}) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with } \mathsf{tail}\}$$



Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

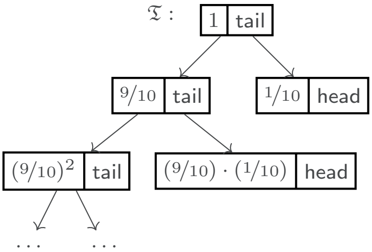
Distribution: $\mu = \{p_1 : t_1, \dots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + (\frac{9}{10})^2 + \dots = \sum_{n=0}^{\infty} (\frac{9}{10})^n = 10$$

Expected Derivation Height:

$$\mathrm{edh}_{\mathcal{P}_{\mathsf{tail}}}(\mathsf{tail}) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with } \mathsf{tail}\} = 10$$



Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

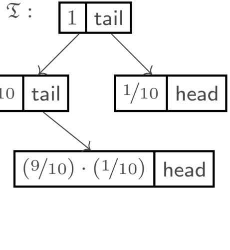
Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + (\frac{9}{10})^2 + \ldots = \sum_{n=0}^{\infty} (\frac{9}{10})^n = 10$$

Expected Derivation Height:

$$\mathrm{edh}_{\mathcal{P}_{\mathsf{tail}}}(\mathsf{tail}) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with } \mathsf{tail}\} = 10$$



Expected Runtime Complexity:

Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\quad \mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

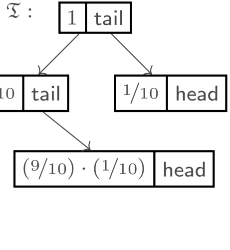
$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \ldots = \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = 10$$

Expected Derivation Height:

$$\mathrm{edh}_{\mathcal{P}_{\mathsf{tail}}}(\mathsf{tail}) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with } \mathsf{tail}\} = 10$$

Expected Runtime Complexity:

$$\mathrm{erc}_{\mathcal{P}_{\mathsf{tail}}}(n) = \sup\{\mathrm{edh}_{\mathcal{P}_{\mathsf{tail}}}(t) \mid t \in \mathcal{T}_B, |t| \le n\}$$



Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

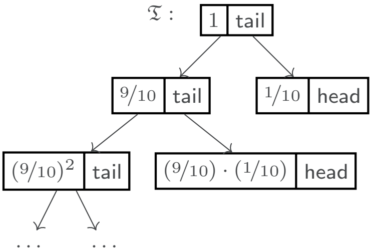
$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + (\frac{9}{10})^2 + \ldots = \sum_{n=0}^{\infty} (\frac{9}{10})^n = 10$$

Expected Derivation Height:

$$\mathrm{edh}_{\mathcal{P}_{\mathsf{tail}}}(\mathsf{tail}) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with } \mathsf{tail}\} = 10$$

Expected Runtime Complexity:

$$\mathrm{erc}_{\mathcal{P}_{\mathsf{tail}}}(n) = \sup\{\mathrm{edh}_{\mathcal{P}_{\mathsf{tail}}}(t) \mid t \in \mathcal{T}_B, |t| \leq n\}$$



Strong Almost-Sure Termination (SAST) [Avanzini & Dal Lago & Yamada'20]

PTRS $\mathcal{P}$ is SAST if $\mathrm{edh}_{\mathcal{P}}(t)$ is finite for every start term t.

Expected Runtime of Probabilistic TRSs

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \{\frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head}\}$ constant complexity: Pol_0

Distribution: $\mu = \{p_1 : t_1, \ldots, p_k : t_k\}$ with $p_1 + \cdots + p_k = 1$

Expected Derivation Length:

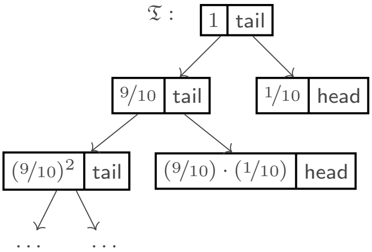
$$\mathrm{edl}(\mathfrak{T}) = 1 + \frac{9}{10} + (\frac{9}{10})^2 + \ldots = \sum_{n=0}^{\infty} (\frac{9}{10})^n = 10$$

Expected Derivation Height:

$$\mathrm{edh}_{\mathcal{P}_{\mathsf{tail}}}(\mathsf{tail}) = \sup\{\mathrm{edl}(\mathfrak{T}) \mid \mathfrak{T} \text{ starts with } \mathsf{tail}\} = 10$$

Expected Runtime Complexity:

$$\mathrm{erc}_{\mathcal{P}_{\mathsf{tail}}}(n) = \sup\{\mathrm{edh}_{\mathcal{P}_{\mathsf{tail}}}(t) \mid t \in \mathcal{T}_B, |t| \leq n\}$$



Strong Almost-Sure Termination (SAST) [Avanzini & Dal Lago & Yamada'20]

PTRS $\mathcal{P}$ is SAST if $\mathrm{edh}_{\mathcal{P}}(t)$ is finite for every start term t.

Proving Almost-Sure Termination

$\mathcal{P}_{\mathsf{tail}}$: $\quad \mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

Proving Almost-Sure Termination

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

Multilinear Polynomials

$x \cdot y$ is multilinear but x^2 is not.

Proving Almost-Sure Termination

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

$\mathcal{I}_{\mathsf{tail}} = 1 \qquad \mathcal{I}_{\mathsf{head}} = 0$

Multilinear Polynomials

$x \cdot y$ is multilinear but x^2 is not.

Proving Almost-Sure Termination

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

$\mathcal{I}_{\mathsf{tail}} = 1 \qquad \mathcal{I}_{\mathsf{head}} = 0$

Multilinear Polynomials

$x \cdot y$ is multilinear but x^2 is not.

Theorem: Natural & Monotonic & Multilinear $\mathcal{I}$ [Kassing & Giesl'24]

For all $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} \in \mathcal{R}$ let

- $\mathit{Pol}(\ell) > \mathit{Pol}(r_j)$ for some $1 \leq j \leq k$
- $\mathit{Pol}(\ell) \geq p_1 \cdot \mathit{Pol}(r_1) + \ldots + p_k \cdot \mathit{Pol}(r_k)$

Then $\mathcal{R}$ is AST.

Proving Almost-Sure Termination

$\mathcal{P}_{\mathsf{tail}}$: $\mathsf{tail} \to \left\{ \frac{9}{10} : \mathsf{tail}, \frac{1}{10} : \mathsf{head} \right\}$

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$\Rightarrow$ proves AST

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Proving Expected Runtime Complexity

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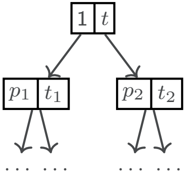

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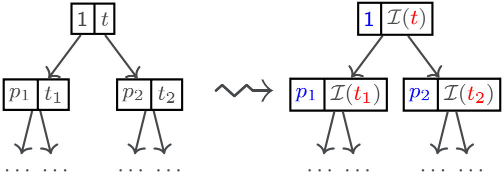

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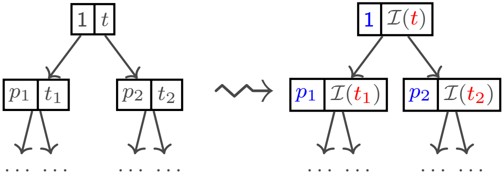

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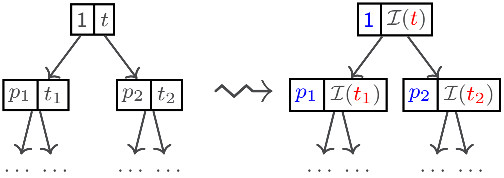

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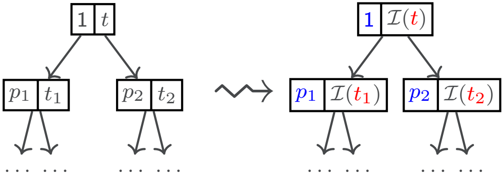

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Proving Expected Runtime Complexity

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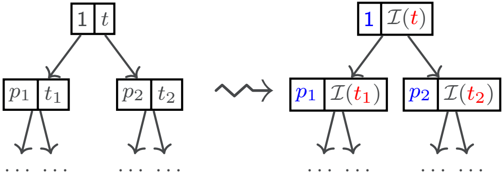

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$> \epsilon \cdot 1 \qquad$ ϵ - minimal decrease for all rules

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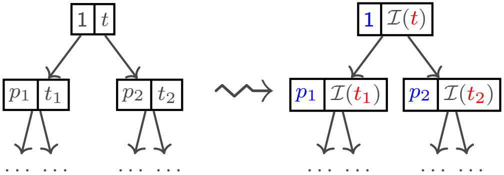

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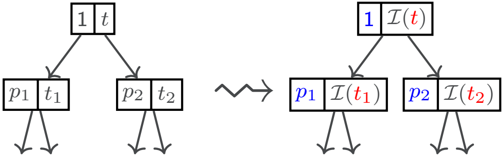

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Goal: Infer expected complexity from the highest degree of $\mathcal{I}$.

Proving Expected Runtime Complexity

$\mathcal{P}_{\text{tail}}$: $1 > \frac{9}{10}$ $\epsilon = {}^{1}/_{10}$

$\mathcal{I}_{\text{tail}} = 1$ $\mathcal{I}_{\text{head}} = 0$ $\rightsquigarrow$ constant complexity: Pol_0

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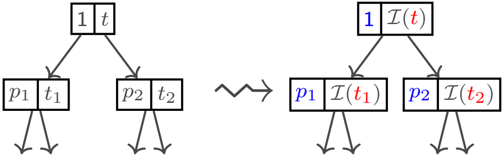

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- Restrict to basic start terms and CPI

Wrap Up (Proving Termination and Complexity of Probabilistic TRSs)

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- **Almost-Sure Termination (AST)**
  - Termination with probability 1.
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  - Finite expected runtime.
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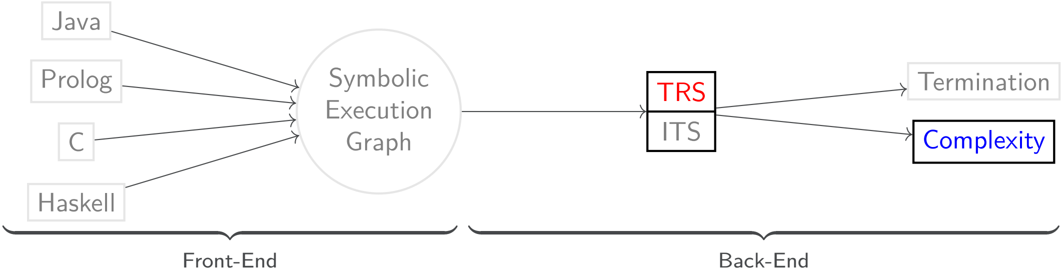
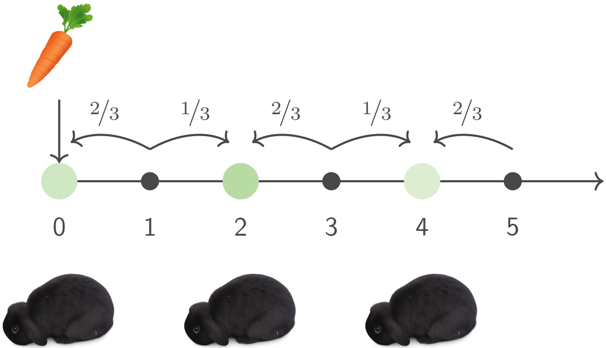
- **Expected Runtime Complexity**
  - Upper bound derived from degree of $\mathcal{I}$ for basic start terms and CPIs.

Termination and Complexity Analysis for Programs



- Proving Termination and Complexity of TRSs
- Proving Termination and Complexity of Probabilistic TRSs
- Disproving Termination of Probabilistic TRSs

Random Walk



$x \leftarrow 3$
while $x > 0$ **do**
$\quad x \leftarrow x - 1 \oplus_{2/3} x \leftarrow x + 1;$

- Does the bunny (program) always reach the carrot (terminate)?
- What is the probability of reaching the carrot (probability of termination)?
- What is the expected number of steps it takes to reach the carrot (expected runtime)?

Characterization of Random Walks

Random Walk μ

A function $\mu : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ s. t. $\mathrm{Supp}(\mu) = \{x \in \mathbb{Z} \mid \mu(x) > 0\}$ is finite and $\sum_{x \in \mathrm{Supp}(\mu)} \mu(x) = 1$.
By $\mathbb{E}(\mu) = \sum_{x \in \mathrm{Supp}(\mu)} x \cdot \mu(x)$ we denote its *expected change*.

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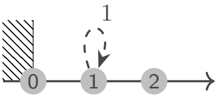

$\mu_1(0) = 1$

Loop Walk μ_1

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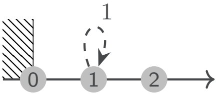

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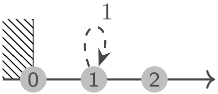

$\mu_1(0) = 1$

Loop Walk μ_1
$(\mu(0) = 1)$
$\Longrightarrow$ not AST and not SAST

Characterization of Random Walks

Random Walk μ

A function $\mu : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ s. t. $\text{Supp}(\mu) = \{x \in \mathbb{Z} \mid \mu(x) > 0\}$ is finite and $\sum_{x \in \text{Supp}(\mu)} \mu(x) = 1$.
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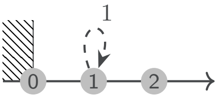

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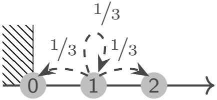

$\mu_2(-1) = 1/3$
$\mu_2(0) = 1/3$
$\mu_2(1) = 1/3$

Symmetric Random Walk μ_2

Characterization of Random Walks

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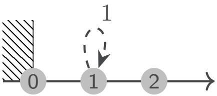

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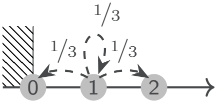

$\mu_2(-1) = 1/3$
$\mu_2(0) = 1/3$
$\mu_2(1) = 1/3$

Symmetric Random Walk μ_2
$(\mu(0) < 1$ and $\mathbb{E}(\mu) = 0)$

Characterization of Random Walks

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A function $\mu : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ s. t. $\mathrm{Supp}(\mu) = \{x \in \mathbb{Z} \mid \mu(x) > 0\}$ is finite and $\sum_{x \in \mathrm{Supp}(\mu)} \mu(x) = 1$.
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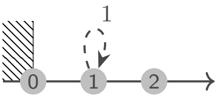

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Loop Walk μ_1
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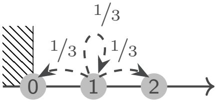

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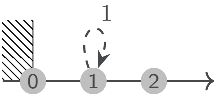

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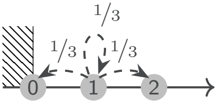

$\mu_2(-1) = 1/3$
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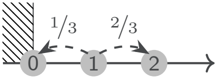
Symmetric Random Walk μ_2
$(\mu(0) < 1$ and $\mathbb{E}(\mu) = 0)$
$\Longrightarrow$ AST and not SAST



$\mu_3(-1) = 1/3$
$\mu_3(1) = 2/3$

Positively Biased Random Walk μ_3

Characterization of Random Walks

Random Walk μ

A function $\mu : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ s. t. $\text{Supp}(\mu) = \{x \in \mathbb{Z} \mid \mu(x) > 0\}$ is finite and $\sum_{x \in \text{Supp}(\mu)} \mu(x) = 1$.
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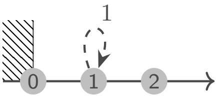
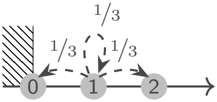


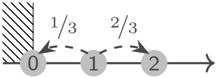
$\mu_1(0) = 1$

Loop Walk μ_1
$(\mu(0) = 1)$
$\Longrightarrow$ not AST and not SAST



$\mu_2(-1) = 1/3$
$\mu_2(0) = 1/3$
$\mu_2(1) = 1/3$

Symmetric Random Walk μ_2
$(\mu(0) < 1$ and $\mathbb{E}(\mu) = 0)$
$\Longrightarrow$ AST and not SAST



$\mu_3(-1) = 1/3$
$\mu_3(1) = 2/3$

Positively Biased Random Walk μ_3
$(\mu(0) < 1$ and $\mathbb{E}(\mu) > 0)$

Characterization of Random Walks

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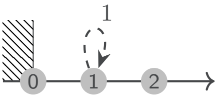
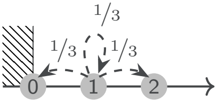


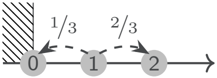
$\mu_1(0) = 1$

Loop Walk μ_1
($\mu(0) = 1$)
$\Longrightarrow$ not AST and not SAST



$\mu_2(-1) = 1/3$
$\mu_2(0) = 1/3$
$\mu_2(1) = 1/3$

Symmetric Random Walk μ_2
($\mu(0) < 1$ and $\mathbb{E}(\mu) = 0$)
$\Longrightarrow$ AST and not SAST



$\mu_3(-1) = 1/3$
$\mu_3(1) = 2/3$

Positively Biased Random Walk μ_3
($\mu(0) < 1$ and $\mathbb{E}(\mu) > 0$)
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Characterization of Random Walks

Random Walk μ

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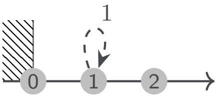
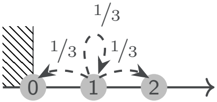


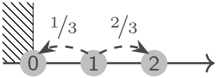
$\mu_1(0) = 1$

Loop Walk μ_1
$(\mu(0) = 1)$
$\implies$ not AST and not SAST



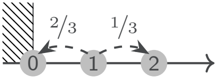
$\mu_2(-1) = 1/3$
$\mu_2(0) = 1/3$
$\mu_2(1) = 1/3$

Symmetric Random Walk μ_2
$(\mu(0) < 1$ and $\mathbb{E}(\mu) = 0)$
$\implies$ AST and not SAST



$\mu_3(-1) = 1/3$
$\mu_3(1) = 2/3$

Positively Biased Random Walk μ_3
$(\mu(0) < 1$ and $\mathbb{E}(\mu) > 0)$
$\implies$ not AST and not SAST



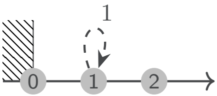
$\mu_4(-1) = 2/3$
$\mu_4(1) = 1/3$

Negatively Biased Random Walk μ_4

Characterization of Random Walks

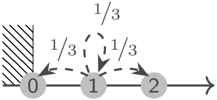
Random Walk μ

A function $\mu : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ s. t. $\mathrm{Supp}(\mu) = \{x \in \mathbb{Z} \mid \mu(x) > 0\}$ is finite and $\sum_{x \in \mathrm{Supp}(\mu)} \mu(x) = 1$.
By $\mathbb{E}(\mu) = \sum_{x \in \mathrm{Supp}(\mu)} x \cdot \mu(x)$ we denote its *expected change*.



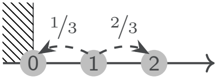
$\mu_1(0) = 1$

Loop Walk μ_1
$(\mu(0) = 1)$
$\implies$ not AST and not SAST



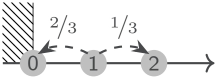
$\mu_2(-1) = 1/3$
$\mu_2(0) = 1/3$
$\mu_2(1) = 1/3$

Symmetric Random Walk μ_2
$(\mu(0) < 1$ and $\mathbb{E}(\mu) = 0)$
$\implies$ AST and not SAST



$\mu_3(-1) = 1/3$
$\mu_3(1) = 2/3$

Positively Biased Random Walk μ_3
$(\mu(0) < 1$ and $\mathbb{E}(\mu) > 0)$
$\implies$ not AST and not SAST



$\mu_4(-1) = 2/3$
$\mu_4(1) = 1/3$

Negatively Biased Random Walk μ_4
$(\mu(0) < 1$ and $\mathbb{E}(\mu) < 0)$

Characterization of Random Walks

Random Walk μ

A function $\mu : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ s. t. $\mathrm{Supp}(\mu) = \{x \in \mathbb{Z} \mid \mu(x) > 0\}$ is finite and $\sum_{x \in \mathrm{Supp}(\mu)} \mu(x) = 1$.
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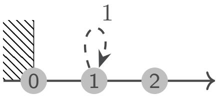
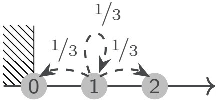


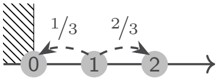
$\mu_1(0) = 1$

Loop Walk μ_1
($\mu(0) = 1$)
$\Longrightarrow$ not AST and not SAST



$\mu_2(-1) = 1/3$
$\mu_2(0) = 1/3$
$\mu_2(1) = 1/3$

Symmetric Random Walk μ_2
($\mu(0) < 1$ and $\mathbb{E}(\mu) = 0$)
$\Longrightarrow$ AST and not SAST



$\mu_3(-1) = 1/3$
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Positively Biased Random Walk μ_3
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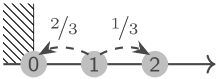
$\Longrightarrow$ not AST and not SAST



$\mu_4(-1) = 2/3$
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Negatively Biased Random Walk μ_4
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Disproving AST and SAST of a PTRS

Disproving Termination of a TRS:

Disproving AST and SAST of a PTRS

Disproving Termination of a TRS: Find t, context C, and substitution σ s. t.

Disproving AST and SAST of a PTRS

Disproving Termination of a TRS: Find t, context C, and substitution σ s. t.

$$t \to C[t\sigma]$$

Disproving AST and SAST of a PTRS

Disproving Termination of a TRS: Find t, context C, and substitution σ s. t.

$$
\begin{array}{c}
t \\
\downarrow \\
C[t\sigma] \\
\downarrow \\
C[C[t\sigma]\sigma] \\
\downarrow \\
\ldots
\end{array}
$$

Disproving AST and SAST of a PTRS

Disproving Termination of a TRS: Find t, context C, and substitution σ s. t.

$$
\begin{array}{c}
t \\
\downarrow \\
C[t\sigma] \\
\downarrow \\
C[C[t\sigma]\sigma] \\
\downarrow \\
\dots
\end{array}
$$

$\mathcal{P}'_{\mathsf{double}}$:

$$
\begin{aligned}
\mathsf{double}(0) &\to 0 \\
\mathsf{double}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x))))
\end{aligned}
$$

Disproving AST and SAST of a PTRS

Disproving Termination of a TRS: Find t, context C, and substitution σ s. t.

$$
\begin{array}{ccc}
t & \qquad & \mathsf{double}(\mathsf{s}(x)) \\
\downarrow & & \downarrow \\
C[t\sigma] & & \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x)))) \\
\downarrow & & \downarrow \\
C[C[t\sigma]\sigma] & & \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(\mathsf{s}(x)))))) \\
\downarrow & & \downarrow \\
\dots & & \dots
\end{array}
$$

$\mathcal{P}'_{\mathsf{double}}$:

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$$

Disproving AST and SAST of a PTRS

Disproving Termination of a TRS: Find t, context C, and substitution σ s. t.

$$
\begin{array}{ccc}
t & \quad & \mathsf{double}(\mathsf{s}(x)) \\
\downarrow & & \downarrow \\
C[t\sigma] & & \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x)))) \\
\downarrow & & \downarrow \\
C[C[t\sigma]\sigma] & & \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(\mathsf{s}(x)))))) \\
\downarrow & & \downarrow \\
\ldots & & \ldots
\end{array}
$$

$\mathcal{P}'_{\mathsf{double}}$:

$$
\begin{aligned}
\mathsf{double}(0) &\to 0 \\
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$$

Disproving AST and SAST of a PTRS

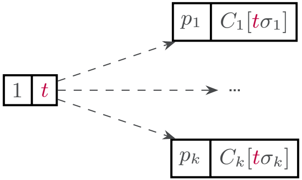
Disproving (S)AST of a PTRS (1.Idea):

Disproving AST and SAST of a PTRS

Disproving (S)AST of a PTRS (1.Idea): Find t, $C_1, \ldots, C_k$, and $\sigma_1, \ldots, \sigma_k$ s. t.

Disproving AST and SAST of a PTRS

Disproving (S)AST of a PTRS (1.Idea): Find t, $C_1, \ldots, C_k$, and $\sigma_1, \ldots, \sigma_k$ s. t.



Disproving AST and SAST of a PTRS

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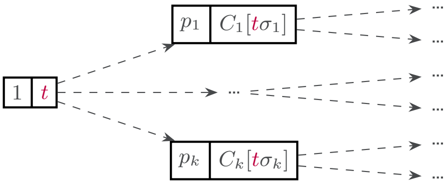

Disproving AST and SAST of a PTRS

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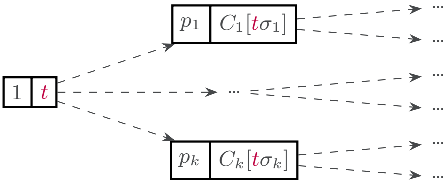
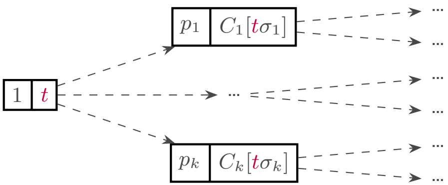


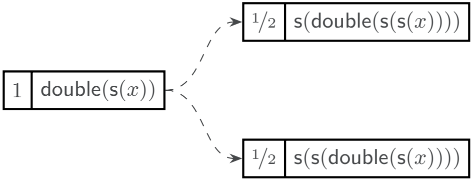
$\mathcal{P}'_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \{1/2 : \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x)))),\ 1/2 : \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(x))))\}$$

Disproving AST and SAST of a PTRS

Disproving (S)AST of a PTRS (1.Idea): Find t, $C_1, \ldots, C_k$, and $\sigma_1, \ldots, \sigma_k$ s. t.





$\mathcal{P}'_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$

$$\mathsf{double}(\mathsf{s}(x)) \to \{1/2 : \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x)))),\ 1/2 : \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(x))))\}$$

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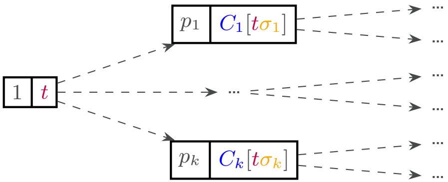
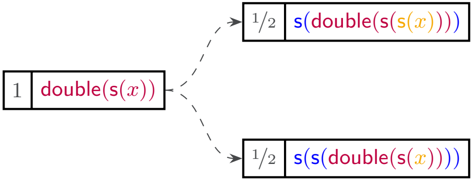
Disproving (S)AST of a PTRS (1.Idea): Find t, $C_1, \ldots, C_k$, and $\sigma_1, \ldots, \sigma_k$ s. t.



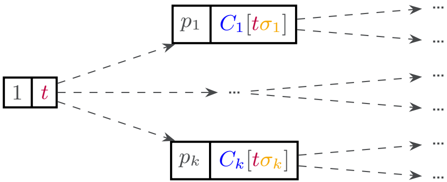


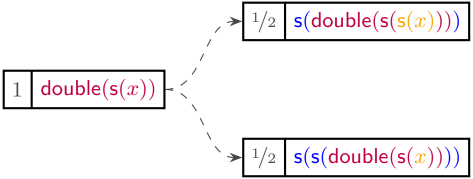
$\mathcal{P}'_{\mathsf{double}}$:

$$\mathsf{double}(0) \to 0$$
$$\mathsf{double}(\mathsf{s}(x)) \to \{1/2 : \mathsf{s}(\mathsf{double}(\mathsf{s}(\mathsf{s}(x)))),\ 1/2 : \mathsf{s}(\mathsf{s}(\mathsf{double}(\mathsf{s}(x))))\}$$

Disproving AST and SAST of a PTRS

Disproving (S)AST of a PTRS (1.Idea): Find t, $C_1, \ldots, C_k$, and $\sigma_1, \ldots, \sigma_k$ s. t.





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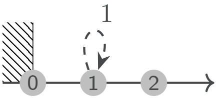
$\Rightarrow$ Embedding loop walks



$\mu_1(0) = 1$

Disproving AST and SAST of a PTRS

Disproving (S)AST of a PTRS (2.Idea): Try to embed symmetric or positively biased random walks

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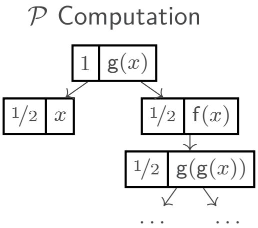

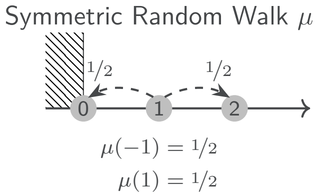

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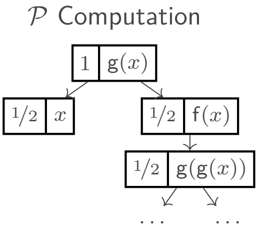

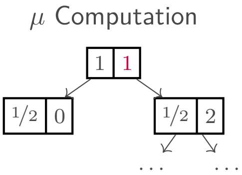

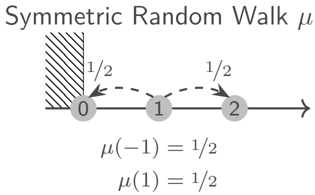

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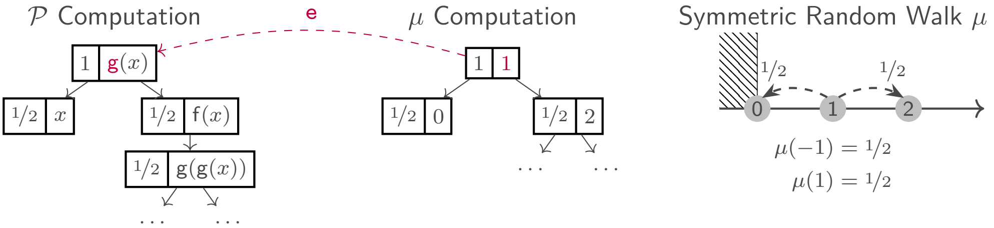

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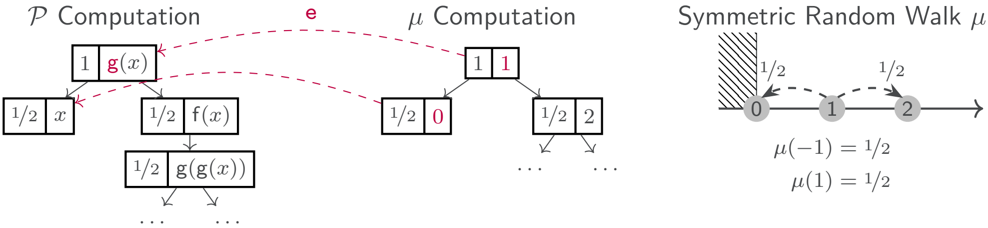

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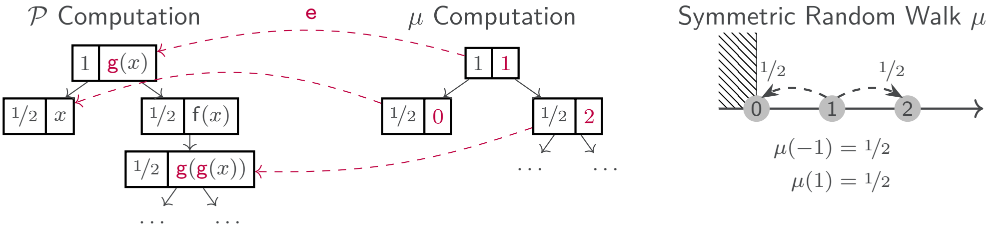

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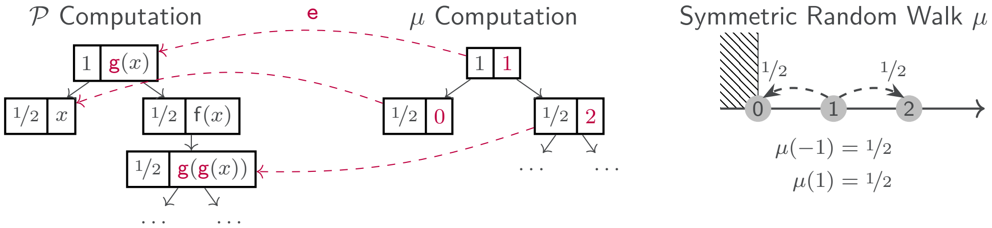

- What does it mean to find a random walk?
  ⇝ Embedding e from computation of μ to computation of $\mathcal{P}$

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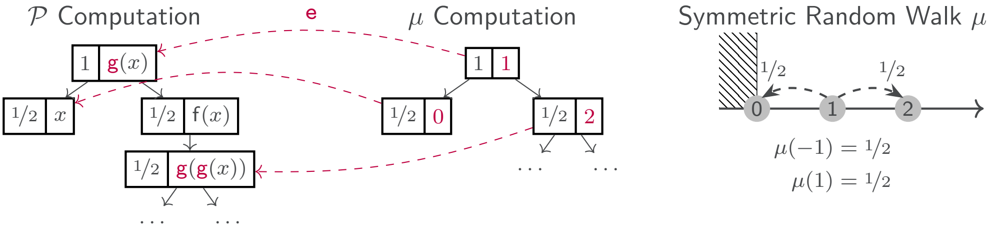

- What does it mean to find a random walk?
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- What and How to Count?

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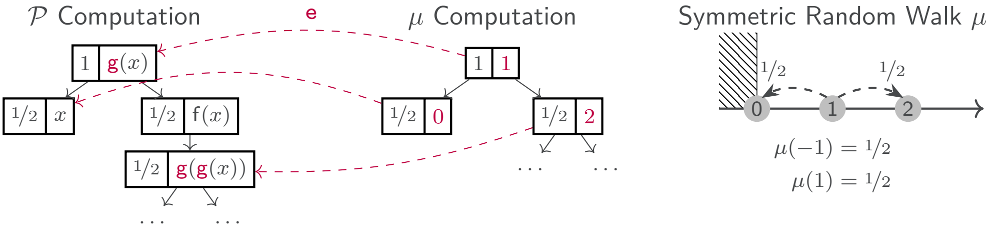

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  $\rightsquigarrow$ Embedding e from computation of μ to computation of $\mathcal{P}$
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Counting Term Occurrences

- Find the maximal number t occurs in s (denoted $\mathrm{maxNO}(t, s)$)

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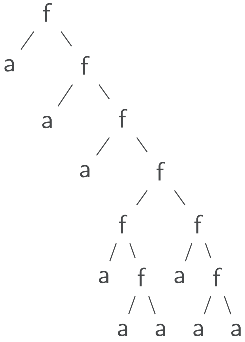
t

s



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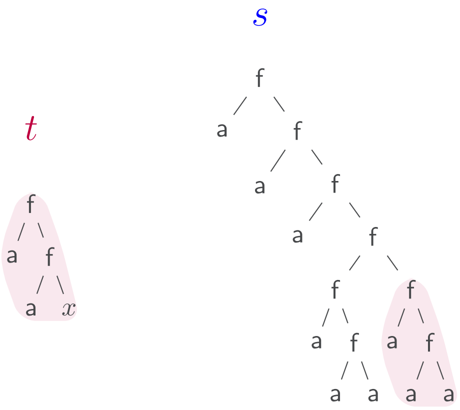

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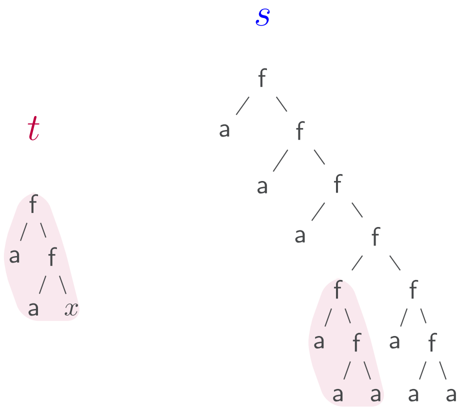

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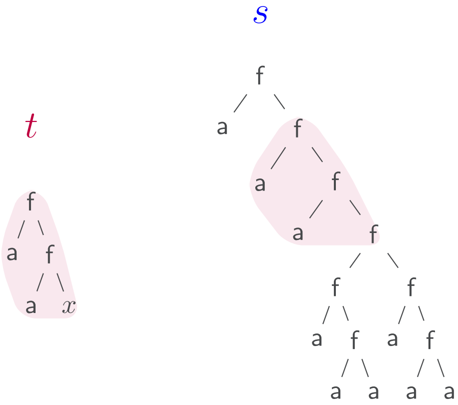

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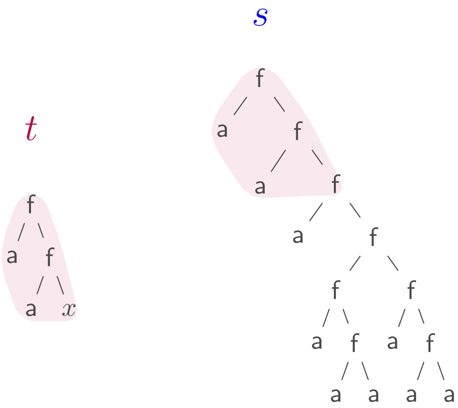

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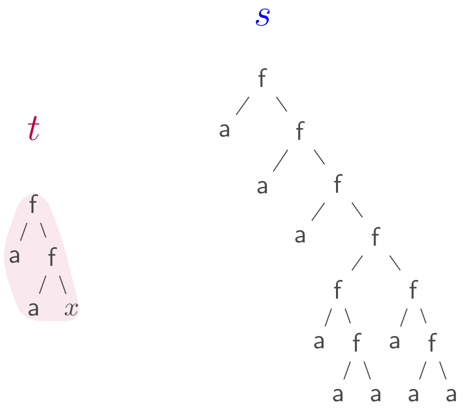
- Only count non-overlapping occurrences!



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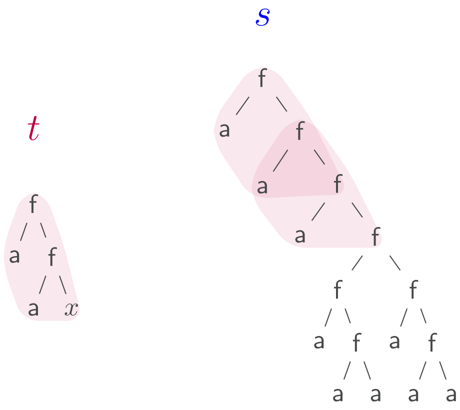

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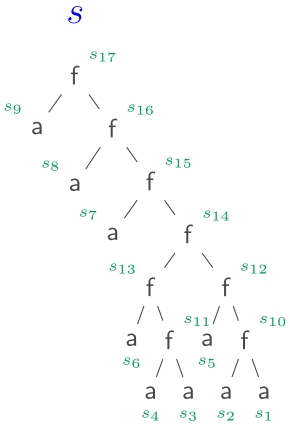
Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$



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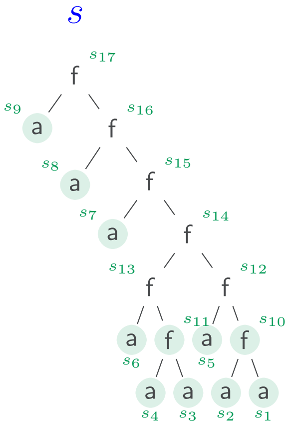
Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$
s_1–s_{11}	0



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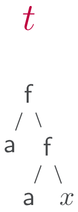
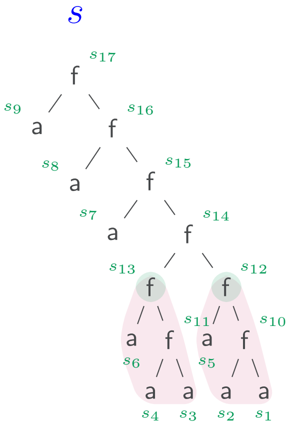
Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$
s_1–s_{11}	0
s_{12}–s_{13}	1





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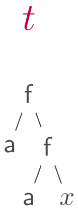
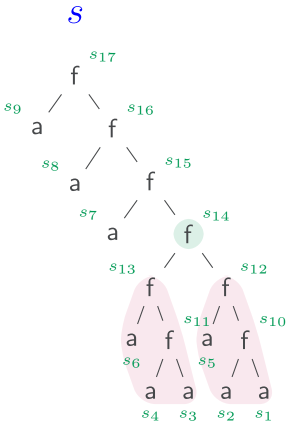
Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$
s_1–s_{11}	0
s_{12}–s_{13}	1
s_{14}	2





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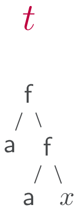
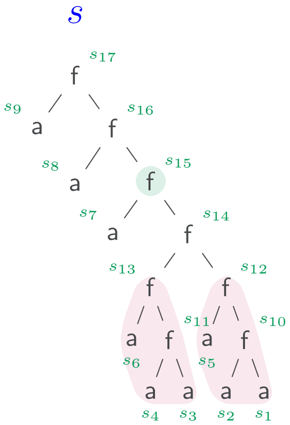
Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$
s_1–s_{11}	0
s_{12}–s_{13}	1
s_{14}	2
s_{15}	2





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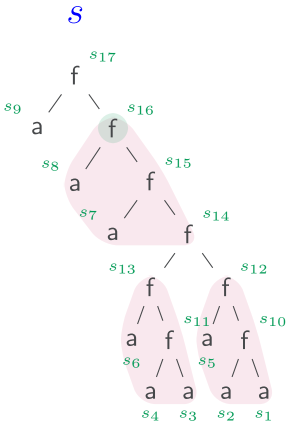
s_1–s_{11}	0
s_{12}–s_{13}	1
s_{14}	2
s_{15}	2
s_{16}	3



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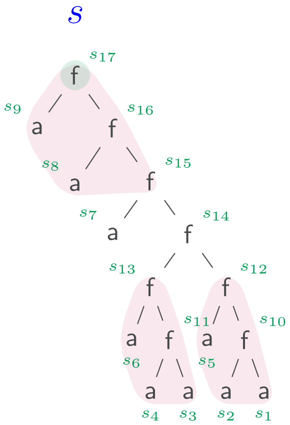
s_1–s_{11}	0
s_{12}–s_{13}	1
s_{14}	2
s_{15}	2
s_{16}	3
s_{17}	3



Counting Term Occurrences

- Find the maximal number t occurs in s (denoted $\mathrm{maxNO}(t, s)$)
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$$\mathrm{maxNO}(t, s) = \alpha_s = \alpha_{s_{17}} = 3$$

Subterm $s' \lhd s$	$\mathrm{maxNO}(t, s')$
s_1–s_{11}	0
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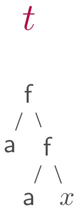
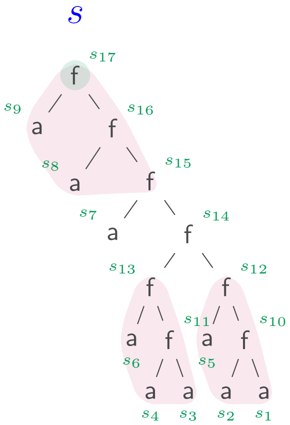
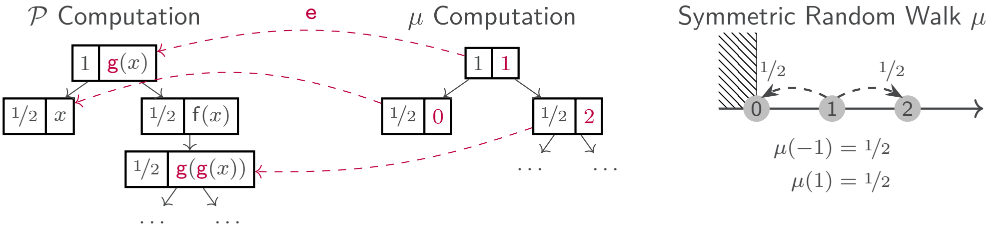
s_{14}	2
s_{15}	2
s_{16}	3
s_{17}	3





Disproving AST and SAST of a PTRS

Disproving (S)AST of a PTRS (2.Idea): Try to find random walks within the computation



- What does it mean to find a random walk?
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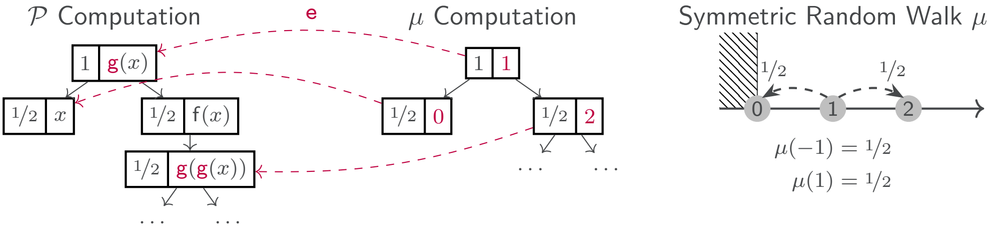

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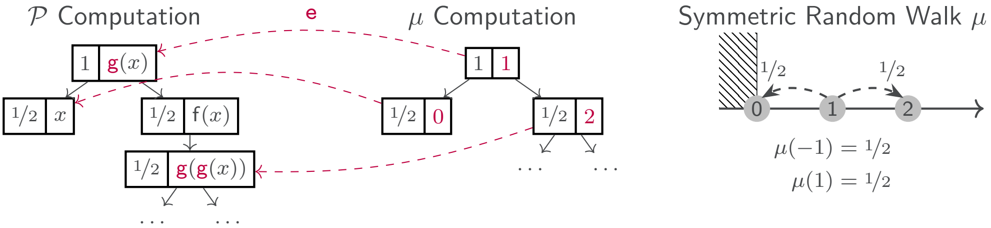

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  $\rightsquigarrow$ Embedding e from computation of μ to computation of $\mathcal{P}$
  - What and How to Count? $\rightsquigarrow$ Count term occurrences
- How to construct the infinite computation? $\rightsquigarrow$ Iteratively rewrite the innermost occurrence

Construct the Infinite Computation

Theorem: Embedding Random Walks [Kassing & Nagel & Schlecht & Giesl (In Review)]

Construct the Infinite Computation

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$\mathcal{P}$ a PTRS

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$\mathcal{P}$ a PTRS, t a linear term

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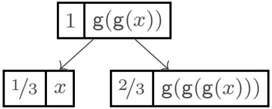
$t = \mathsf{g}(\mathsf{g}(x))$

Computation $\mathfrak{T}$



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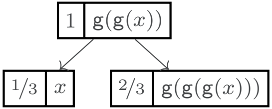
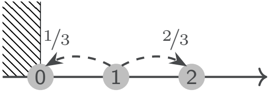


Positively Biased Random Walk μ



$\mu(-1) = 1/3$

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Why do we need non-overlapping occurrences?

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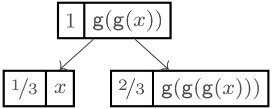
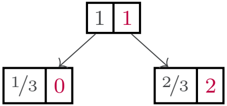
$$\mathsf{g}(\mathsf{g}(x)) \to \{1/3 : x, 2/3 : \mathsf{g}(\mathsf{g}(\mathsf{g}(x)))\}$$

$t = \mathsf{g}(\mathsf{g}(x))$

Computation $\mathfrak{T}$



μ Computation



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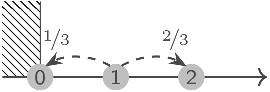

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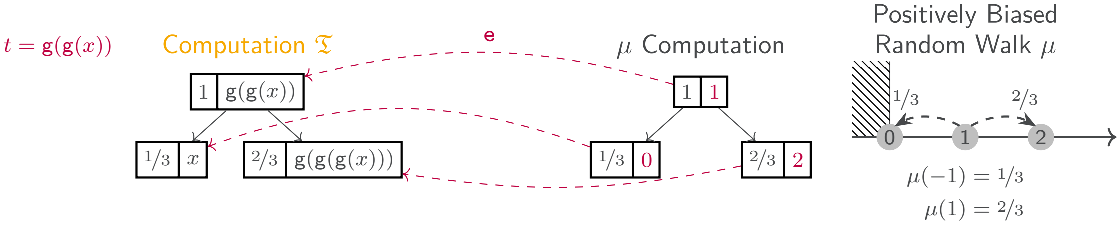

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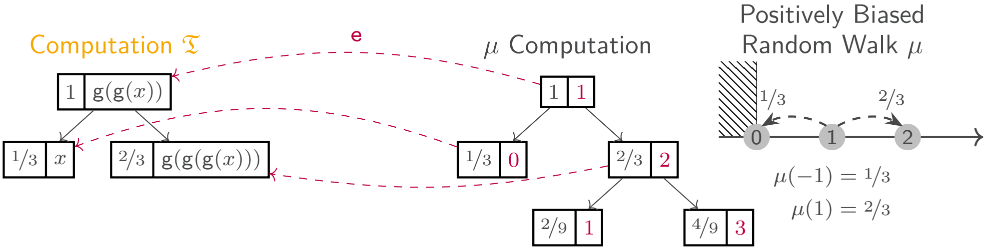

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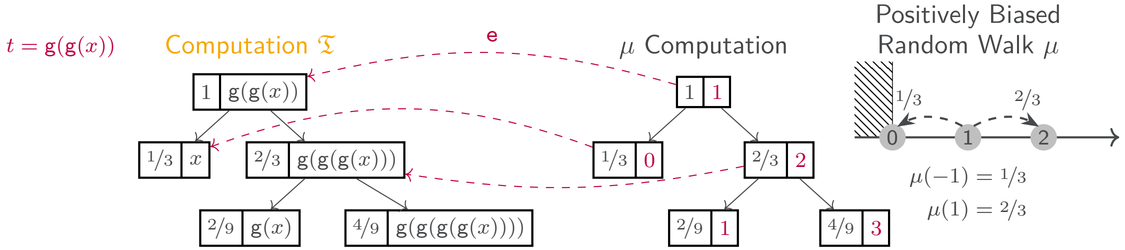

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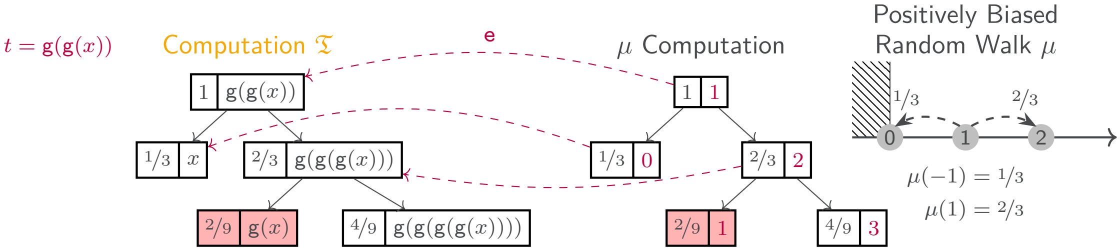

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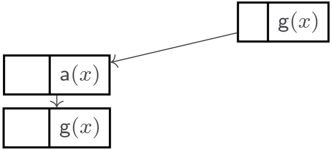

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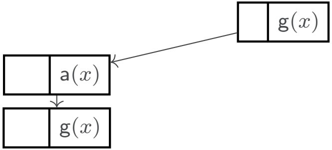

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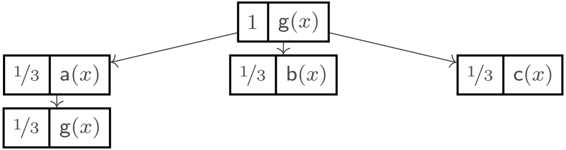

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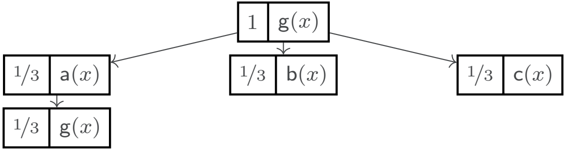

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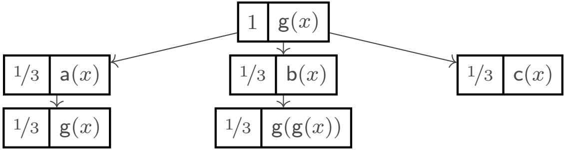

Conclusion

1. What are applications of probabilistic programs? Why are they interesting?
   - Monte Carlo and Las Vegas algorithms
   - Increase the expected worst-case runtime (prevent adversarial attacks)

2. How to analyze probabilistic programs?
   - Transform to a simpler backend language (term rewriting)
   - Prove $\mathtt{AST}$ and $\mathtt{SAST}$ via ranking functions
   - Derive upper expected runtime bounds via ranking functions
   - Disprove $\mathtt{AST}$ and $\mathtt{SAST}$ via embeddings of random walks

   Further techniques:

   - Modularize proofs via dependency pairs
   - Transform the rules to simplify proofs
   - ...

UnRAVeL Symposium 2026

UnRAVeL – UNcertainty and Randomness in Algorithms, VErification and Logic

Date: May 26–29, 2026
Location: RWTH Aachen University, Aachen
Topics: Discrete Structures, Verification, Control, Railway, Combinatorial Optimization, and more!

Keynote Speakers:

- Michal Pilipczuk (University of Warsaw)
- Rupak Majumdar (MPI-SWS)
- Maurice Heemels (TU Eindhoven)
- Francesca Parise (Cornell University)
- Rico Zenklusen (ETH Zürich)
- And further talks by UnRAVeL members and alumni

More information: `https://www.unravel.rwth-aachen.de`
(**Registration (including dinner) is free!**)

Implementation

- Fully implemented in AProVE
- Evaluated on 158 benchmarks

Category	AProVE
AST	70
$\neg$ AST	24
Unknown	68

Category	AProVE
SAST	49
$\neg$ SAST	31
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