Decidability of AST for certain classes of PTRSs

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Research Group Computer Science 2
“Programming Languages and Verification”

June 2023

Termination and Complexity Analysis for Programs

`Java`

`C`

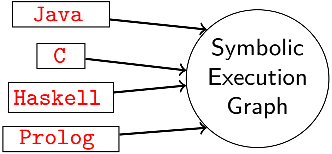
`Haskell`

`Prolog`

Termination

Complexity

Termination and Complexity Analysis for Programs



`Java`

`C`

`Haskell`

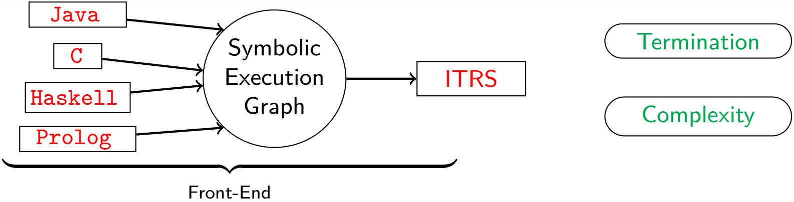
`Prolog`

Symbolic Execution Graph

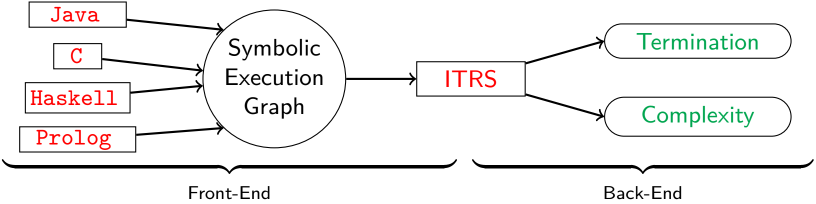
Termination

Complexity

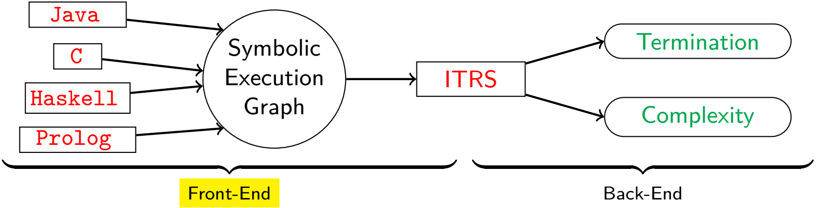
Termination and Complexity Analysis for Programs



Termination and Complexity Analysis for Programs

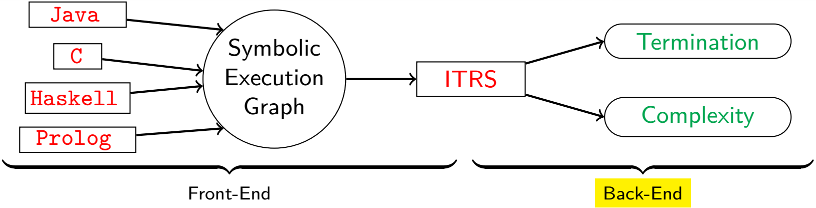


Termination and Complexity Analysis for Programs



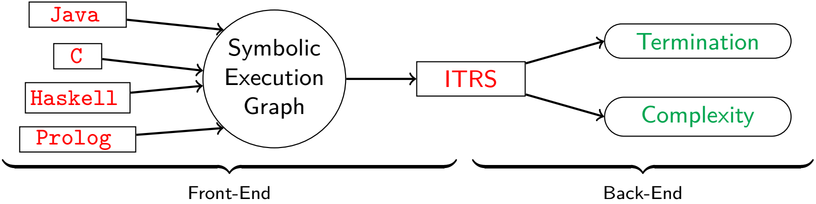
- language-specific features when generating symbolic execution graph

Termination and Complexity Analysis for Programs



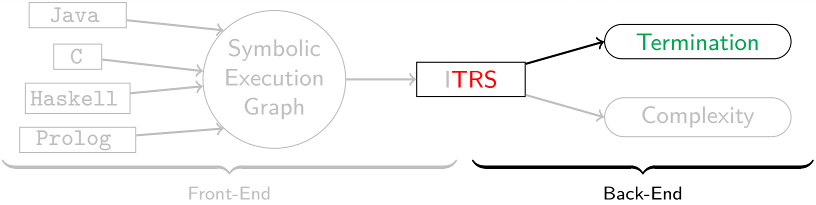
- language-specific features when generating symbolic execution graph
- back-end analyzes Term Rewrite Systems and/or Integer Transition Systems

Termination and Complexity Analysis for Programs

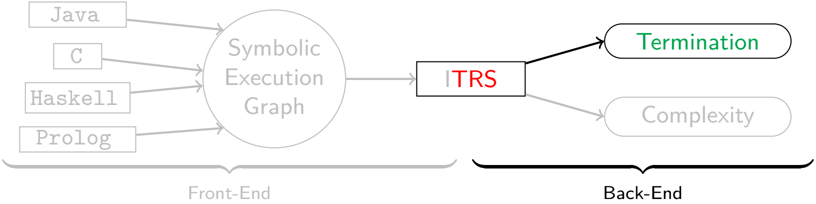


- language-specific features when generating symbolic execution graph
- back-end analyzes Term Rewrite Systems and/or Integer Transition Systems
- powerful termination and complexity analysis implemented in AProVE
  - Termination Competition since 2004 (`Java`, `C`, `Haskell`, `Prolog`, `TRS`)
  - SV-COMP since 2014 (`C`)

Termination of Term Rewrite Systems

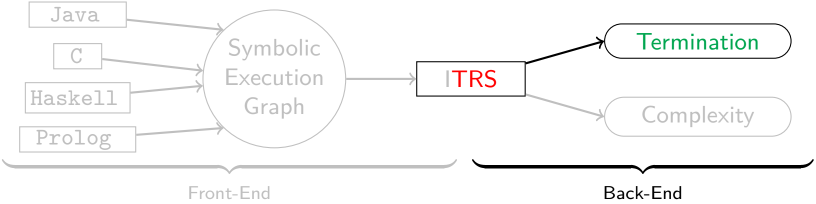


Termination of Term Rewrite Systems



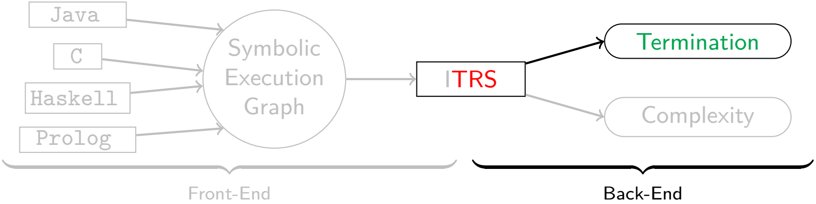
1. Termination Analysis for TRSs

Termination of Term Rewrite Systems



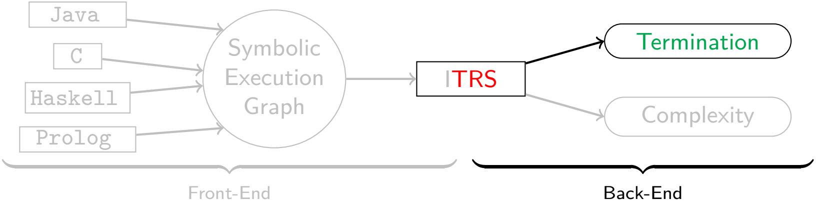
1. Termination Analysis for TRSs
2. Dependency Pairs for TRSs

Termination of Term Rewrite Systems



1. Termination Analysis for TRSs
2. Dependency Pairs for TRSs
3. Termination and Dependency Pairs for *Probabilistic* TRSs

Termination of Term Rewrite Systems



Java

C

Haskell

Prolog

Symbolic Execution Graph

ITRS

Termination

Complexity

Front-End

Back-End

1. Termination Analysis for TRSs
2. Dependency Pairs for TRSs
3. Termination and Dependency Pairs for *Probabilistic* TRSs

Termination of TRSs

Termination of TRSs

$$\mathcal{R}_{bool}: \quad \begin{aligned} \mathsf{and}(\mathsf{true}, \mathsf{true}) &\to \mathsf{true} \\ \mathsf{and}(x, \mathsf{false}) &\to \mathsf{false} \\ \mathsf{and}(\mathsf{false}, x) &\to \mathsf{and}(\mathsf{true}, \mathsf{not}(\mathsf{true})) \\ \mathsf{not}(\mathsf{false}) &\to \mathsf{true} \\ \mathsf{not}(\mathsf{true}) &\to \mathsf{and}(\mathsf{false}, \mathsf{false}) \end{aligned}$$

Termination of TRSs

$$\mathcal{R}_{bool}: \quad \begin{array}{rcl} \mathsf{and}(\mathsf{true},\mathsf{true}) & \to & \mathsf{true} \\ \mathsf{and}(x,\mathsf{false}) & \to & \mathsf{false} \\ \mathsf{and}(\mathsf{false},x) & \to & \mathsf{and}(\mathsf{true},\mathsf{not}(\mathsf{true})) \\ \mathsf{not}(\mathsf{false}) & \to & \mathsf{true} \\ \mathsf{not}(\mathsf{true}) & \to & \mathsf{and}(\mathsf{false},\mathsf{false}) \end{array}$$

$\mathsf{a}(\mathsf{f},\mathsf{t})$

Termination of TRSs

$$\mathcal{R}_{bool}: \quad \begin{array}{rcl} \mathsf{and}(\mathsf{true},\mathsf{true}) & \to & \mathsf{true} \\ \mathsf{and}(x,\mathsf{false}) & \to & \mathsf{false} \\ \mathsf{and}(\mathsf{false},x) & \to & \mathsf{and}(\mathsf{true},\mathsf{not}(\mathsf{true})) \\ \mathsf{not}(\mathsf{false}) & \to & \mathsf{true} \\ \mathsf{not}(\mathsf{true}) & \to & \mathsf{and}(\mathsf{false},\mathsf{false}) \end{array}$$

$$\mathsf{a}(\mathsf{f},\mathsf{t}) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t},\mathsf{n}(\mathsf{t}))$$

Termination of TRSs

$$\mathcal{R}_{bool}: \quad \begin{array}{rcl} \mathsf{and}(\mathsf{true}, \mathsf{true}) & \to & \mathsf{true} \\ \mathsf{and}(x, \mathsf{false}) & \to & \mathsf{false} \\ \mathsf{and}(\mathsf{false}, x) & \to & \mathsf{and}(\mathsf{true}, \mathsf{not}(\mathsf{true})) \\ \mathsf{not}(\mathsf{false}) & \to & \mathsf{true} \\ \mathsf{not}(\mathsf{true}) & \to & \mathsf{and}(\mathsf{false}, \mathsf{false}) \end{array}$$

$$\mathsf{a}(\mathsf{f}, \mathsf{t}) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{n}(\mathsf{t})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{a}(\mathsf{f}, \mathsf{f}))$$

Termination of TRSs

$$\mathcal{R}_{bool}: \quad \begin{array}{rcl} \mathsf{and}(\mathsf{true}, \mathsf{true}) & \to & \mathsf{true} \\ \mathsf{and}(x, \mathsf{false}) & \to & \mathsf{false} \\ \mathsf{and}(\mathsf{false}, x) & \to & \mathsf{and}(\mathsf{true}, \mathsf{not}(\mathsf{true})) \\ \mathsf{not}(\mathsf{false}) & \to & \mathsf{true} \\ \mathsf{not}(\mathsf{true}) & \to & \mathsf{and}(\mathsf{false}, \mathsf{false}) \end{array}$$

$$\mathsf{a}(\mathsf{f}, \mathsf{t}) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{n}(\mathsf{t})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{a}(\mathsf{f}, \mathsf{f})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{f})$$

Termination of TRSs

$$
\mathcal{R}_{bool}: \quad \begin{array}{rcl}
\mathsf{and}(\mathsf{true}, \mathsf{true}) & \to & \mathsf{true} \\
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\end{array}
$$

$$
\mathsf{a}(\mathsf{f}, \mathsf{t}) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{n}(\mathsf{t})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{a}(\mathsf{f}, \mathsf{f})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{f}) \to_{\mathcal{R}_{bool}} \mathsf{f}
$$

Termination of TRSs

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$$\mathsf{a}(\mathsf{f},\mathsf{t}) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t},\mathsf{n}(\mathsf{t})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t},\mathsf{a}(\mathsf{f},\mathsf{f})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t},\mathsf{f}) \to_{\mathcal{R}_{bool}} \mathsf{f}$$

Termination

$\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \ldots$

Termination of TRSs

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Termination

$\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \dots$

$\mathsf{a}(\mathsf{f},\mathsf{t})$

Termination of TRSs

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$\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \dots$

$\mathsf{a}(\mathsf{f},\mathsf{t}) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t},\mathsf{n}(\mathsf{t}))$

Termination of TRSs

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$\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \ldots$

$$\mathsf{a}(\mathsf{f},\mathsf{t}) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t},\mathsf{n}(\mathsf{t})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t},\mathsf{a}(\mathsf{f},\mathsf{f}))$$

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Termination

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Termination of TRSs

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Termination

$\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \dots$

$$\mathsf{a}(\mathsf{f},\mathsf{t}) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t},\mathsf{n}(\mathsf{t})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t},\mathsf{a}(\mathsf{f},\mathsf{f})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t},\mathsf{a}(\mathsf{t},\mathsf{n}(\mathsf{t}))) \to_{\mathcal{R}_{bool}} \dots$$

Termination of TRSs

$$\mathcal{R}_{bool}: \quad \begin{array}{rcl} \mathsf{and}(\mathsf{true}, \mathsf{true}) & \to & \mathsf{true} \\ \mathsf{and}(x, \mathsf{false}) & \to & \mathsf{false} \\ \mathsf{and}(\mathsf{false}, x) & \to & \mathsf{and}(\mathsf{true}, \mathsf{not}(\mathsf{true})) \\ \mathsf{not}(\mathsf{false}) & \to & \mathsf{true} \\ \mathsf{not}(\mathsf{true}) & \to & \mathsf{and}(\mathsf{false}, \mathsf{false}) \end{array}$$

$$\mathsf{a}(\mathsf{f}, \mathsf{t}) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{n}(\mathsf{t})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{a}(\mathsf{f}, \mathsf{f})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{f}) \to_{\mathcal{R}_{bool}} \mathsf{f}$$

Termination

$\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \ldots$

$$\mathsf{a}(\mathsf{f}, \mathsf{t}) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{n}(\mathsf{t})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{a}(\mathsf{f}, \mathsf{f})) \to_{\mathcal{R}_{bool}} \mathsf{a}(\mathsf{t}, \mathsf{a}(\mathsf{t}, \mathsf{n}(\mathsf{t}))) \to_{\mathcal{R}_{bool}} \ldots$$

$\Rightarrow \mathcal{R}_{bool}$ is not terminating

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $$\mathrm{g}(\mathcal{O}) \quad \to \quad \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{2} : \mathrm{g}(\mathrm{g}(\mathcal{O})) \,\}$$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\mathsf{g}(\mathcal{O}) \to \{\, {}^1\!/\!_2 : \mathcal{O},\ {}^1\!/\!_2 : \mathsf{g}(\mathsf{g}(\mathcal{O}))\,\}$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

Termination of Probabilistic TRSs

$$\mathcal{R}_{rw}: \quad \mathsf{g}(\mathcal{O}) \to \{\, 1/2 : \mathcal{O},\ 1/2 : \mathsf{g}(\mathsf{g}(\mathcal{O}))\,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\{\, 1 : \mathsf{g}(\mathcal{O}) \,\}$$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\mathsf{g}(\mathcal{O}) \to \{\, {}^1\!/_2 : \mathcal{O},\ {}^1\!/_2 : \mathsf{g}(\mathsf{g}(\mathcal{O}))\,\}$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\begin{aligned} & \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\ \rightrightarrows_{\mathcal{R}_{rw}} \; & \{\, {}^1\!/_2 : \mathcal{O},\ {}^1\!/_2 : \mathsf{g}^2(\mathcal{O}) \,\} \end{aligned}$$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $$\mathsf{g}(\mathcal{O}) \to \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{2} : \mathsf{g}(\mathsf{g}(\mathcal{O}))\,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\begin{aligned} & \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\ \rightrightarrows_{\mathcal{R}_{rw}} \ & \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{2} : \mathsf{g}^{2}(\mathcal{O}) \,\} \\ \rightrightarrows_{\mathcal{R}_{rw}} \ & \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{4} : \mathsf{g}(\mathcal{O}),\ {}^{1}\!/_{4} : \mathsf{g}^{3}(\mathcal{O}) \,\} \end{aligned}$$

Termination of Probabilistic TRSs

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Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\begin{aligned}
& \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{2} : \mathsf{g}^2(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{4} : \mathsf{g}(\mathcal{O}),\ {}^{1}\!/_{4} : \mathsf{g}^3(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{8} : \mathcal{O},\ {}^{1}\!/_{8} : \mathsf{g}^2(\mathcal{O}),
\end{aligned}$$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $$\mathrm{g}(\mathcal{O}) \to \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{2} : \mathrm{g}(\mathrm{g}(\mathcal{O}))\,\}$$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\begin{aligned}
& \{\, 1 : \mathrm{g}(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{2} : \mathrm{g}^2(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{4} : \mathrm{g}(\mathcal{O}),\ {}^{1}\!/_{4} : \mathrm{g}^3(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, {}^{1}\!/_{2} : \mathcal{O},\ {}^{1}\!/_{8} : \mathcal{O},\ {}^{1}\!/_{8} : \mathrm{g}^2(\mathcal{O}),\ {}^{1}\!/_{8} : \mathrm{g}^2(\mathcal{O}),\ {}^{1}\!/_{8} : \mathrm{g}^4(\mathcal{O}) \,\}
\end{aligned}$$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\quad \mathsf{g}(\mathcal{O}) \to \{\, ^1\!/_2 : \mathcal{O},\ ^1\!/_2 : \mathsf{g}(\mathsf{g}(\mathcal{O}))\,\}$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\begin{aligned}
& \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, ^1\!/_2 : \mathcal{O},\ ^1\!/_2 : \mathsf{g}^2(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, ^1\!/_2 : \mathcal{O},\ ^1\!/_4 : \mathsf{g}(\mathcal{O}),\ ^1\!/_4 : \mathsf{g}^3(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, ^1\!/_2 : \mathcal{O},\ ^1\!/_8 : \mathcal{O},\ ^1\!/_8 : \mathsf{g}^2(\mathcal{O}),\ ^1\!/_8 : \mathsf{g}^2(\mathcal{O}),\ ^1\!/_8 : \mathsf{g}^4(\mathcal{O}) \,\}
\end{aligned}$$

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\mathsf{g}(\mathcal{O}) \to \{\, ^1/_2 : \mathcal{O},\ ^1/_2 : \mathsf{g}(\mathsf{g}(\mathcal{O}))\,\}$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\begin{aligned} & \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\ \rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, ^1/_2 : \mathcal{O},\ ^1/_2 : \mathsf{g}^2(\mathcal{O}) \,\} \\ \rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, ^1/_2 : \mathcal{O},\ ^1/_4 : \mathsf{g}(\mathcal{O}),\ ^1/_4 : \mathsf{g}^3(\mathcal{O}) \,\} \\ \rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, ^1/_2 : \mathcal{O},\ ^1/_8 : \mathcal{O},\ ^1/_8 : \mathsf{g}^2(\mathcal{O}),\ ^1/_8 : \mathsf{g}^2(\mathcal{O}),\ ^1/_8 : \mathsf{g}^4(\mathcal{O}) \,\} \end{aligned}$$

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Termination of Probabilistic TRSs

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Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$
\begin{aligned}
& \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, 1/2 : \mathcal{O},\ 1/2 : \mathsf{g}^2(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, 1/2 : \mathcal{O},\ 1/4 : \mathsf{g}(\mathcal{O}),\ 1/4 : \mathsf{g}^3(\mathcal{O}) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \quad & \{\, 1/2 : \mathcal{O},\ 1/8 : \mathcal{O},\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^4(\mathcal{O}) \,\}
\end{aligned}
$$

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$ No
- $\mathcal{R}$ is *almost-surely terminating (AST)*
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Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\mathsf{g}(\mathcal{O}) \to \{\, 1/2 : \mathcal{O},\ 1/2 : \mathsf{g}(\mathsf{g}(\mathcal{O}))\,\}$

Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$|\mu|$

$$\{\, 1 : \mathsf{g}(\mathcal{O}) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, 1/2 : \mathcal{O},\ 1/2 : \mathsf{g}^2(\mathcal{O}) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, 1/2 : \mathcal{O},\ 1/4 : \mathsf{g}(\mathcal{O}),\ 1/4 : \mathsf{g}^3(\mathcal{O}) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, 1/2 : \mathcal{O},\ 1/8 : \mathcal{O},\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^2(\mathcal{O}),\ 1/8 : \mathsf{g}^4(\mathcal{O}) \,\}$$

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Termination of Probabilistic TRSs

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Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$|\mu|$

$$\{\, 1 : \mathsf{g}(\mathcal{O}) \,\} \qquad 0$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, ^1\!/_2 : \mathcal{O},\ ^1\!/_2 : \mathsf{g}^2(\mathcal{O}) \,\}$$

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Termination of Probabilistic TRSs

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$$\begin{array}{llr}
 & & |\mu| \\
 & \{\, 1 : \mathsf{g}(\mathcal{O}) \,\} & 0 \\
\rightrightarrows_{\mathcal{R}_{rw}} & \{\, 1/2 : \mathcal{O},\; 1/2 : \mathsf{g}^2(\mathcal{O}) \,\} & 1/2 \\
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\rightrightarrows_{\mathcal{R}_{rw}} & \{\, 1/2 : \mathcal{O},\; 1/8 : \mathcal{O},\; 1/8 : \mathsf{g}^2(\mathcal{O}),\; 1/8 : \mathsf{g}^2(\mathcal{O}),\; 1/8 : \mathsf{g}^4(\mathcal{O}) \,\} &
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\rightrightarrows_{\mathcal{R}_{rw}} & \{\, {}^1\!/_2 : \mathcal{O},\ {}^1\!/_2 : \mathsf{g}^2(\mathcal{O}) \,\} & {}^1\!/_2 \\
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$$
\begin{array}{lll}
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\rightrightarrows_{\mathcal{R}_{rw}} & \{\, {}^1\!/\!_2 : \mathcal{O},\ {}^1\!/\!_2 : \mathsf{g}^2(\mathcal{O}) \,\} & {}^1\!/\!_2 \\
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Right-Ground Termination is Decidable

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Right-Ground TRS

$\mathcal{R}$ is right-ground if for every rule $\ell \to r \in \mathcal{R}$ the rhs r is a ground term, i.e., contains no variables.

Right-Ground Termination is Decidable

Right-Ground TRS

$\mathcal{R}$ is right-ground if for every rule $\ell \to r \in \mathcal{R}$ the rhs r is a ground term, i.e., contains no variables.

$$\mathcal{R}_{bool}: \quad \begin{aligned} \mathsf{and}(\mathsf{true}, \mathsf{true}) &\to \mathsf{true} \\ \mathsf{and}(x, \mathsf{false}) &\to \mathsf{false} \\ \mathsf{and}(\mathsf{false}, x) &\to \mathsf{and}(\mathsf{true}, \mathsf{not}(\mathsf{true})) \\ \mathsf{not}(\mathsf{false}) &\to \mathsf{true} \\ \mathsf{not}(\mathsf{true}) &\to \mathsf{and}(\mathsf{false}, \mathsf{false}) \end{aligned}$$

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$$
\mathcal{R}_{bool}: \qquad
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\mathsf{and}(\mathsf{true}, \mathsf{true}) & \to & \mathsf{true} \\
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\mathsf{and}(\mathsf{false}, x) & \to & \mathsf{and}(\mathsf{true}, \mathsf{not}(\mathsf{true})) \\
\mathsf{not}(\mathsf{false}) & \to & \mathsf{true} \\
\mathsf{not}(\mathsf{true}) & \to & \mathsf{and}(\mathsf{false}, \mathsf{false})
\end{array}
$$

$\Rightarrow \mathcal{R}_{bool}$ is right-ground.

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$\Rightarrow \mathcal{R}_{bool}$ is right-ground.

Theorem: Right-Ground Termination

Termination for the class of right-ground TRSs is decidable.

Right-Ground Termination Idea

Right-Ground Termination Idea

Lemma: Right-Ground Termination

Let $\mathcal{R}$ be a right-ground TRS. $\mathcal{R}$ is terminating iff. there exists no rule $\ell \to r \in \mathcal{R}$ such that $r \to_{\mathcal{R}}^{+} C[r]$ for some context C.

Right-Ground Termination Idea

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$\Rightarrow$:

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$\Rightarrow$:Assume that there exists a rule $\ell \to r \in \mathcal{R}$ such that $r \to_{\mathcal{R}}^{+} C[r]$

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Lemma: Right-Ground Termination

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$\Rightarrow$:Assume that there exists a rule $\ell \to r \in \mathcal{R}$ such that $r \to_{\mathcal{R}}^{+} C[r]$, then

$$r \to_{\mathcal{R}}^{+} C[r] \to_{\mathcal{R}}^{+} C[C[r]] \to_{\mathcal{R}}^{+} \ldots$$

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$\Leftarrow$:

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$\Leftarrow$:Assume that $\mathcal{R}$ is not terminating, then there exists

$$t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \dots$$

Right-Ground Termination Idea

Lemma: Right-Ground Termination

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$$t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \dots$$

Proof by induction of $|\mathcal{R}|$

Algorithm for Right-Ground Termination

Algorithm:

Algorithm for Right-Ground Termination

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1. For $\mathcal{R} = \{\ell_1 \to r_1, \ldots, \ell_n \to r_n\}$ let $T_i = \{r_i\}$, $1 \leq i \leq n$

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$\mathcal{R}_1$:

$$\begin{array}{rcl} \mathsf{f}(\mathsf{a},\mathsf{b}) & \to & \mathsf{f}(\mathsf{a},\mathsf{g}) \\ \mathsf{f}(\mathsf{b},x) & \to & \mathsf{f}(\mathsf{g},\mathsf{b}) \\ \mathsf{g} & \to & \mathsf{a} \end{array}$$

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$\mathsf{f}(\mathsf{a},\mathsf{g})$

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$$\begin{tikzcd} \mathsf{f}(\mathsf{a},\mathsf{g}) \arrow[d] & \mathsf{f}(\mathsf{g},\mathsf{b}) \\ \mathsf{f}(\mathsf{a},\mathsf{a}) & \end{tikzcd}$$

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$$\begin{array}{ccc} \mathsf{f}(\mathsf{a},\mathsf{g}) & \mathsf{f}(\mathsf{g},\mathsf{b}) & \mathsf{a} \\ \downarrow & \downarrow & \\ \mathsf{f}(\mathsf{a},\mathsf{a}) & \mathsf{f}(\mathsf{a},\mathsf{b}) & \\ & \downarrow & \\ & \mathsf{f}(\mathsf{a},\mathsf{g}) & \\ & \downarrow & \\ & \mathsf{f}(\mathsf{a},\mathsf{a}) & \end{array}$$

Algorithm for Right-Ground Termination

Algorithm:

1. For $\mathcal{R} = \{\ell_1 \to r_1, \dots, \ell_n \to r_n\}$ let $T_i = \{r_i\}$, $1 \leq i \leq n$
2. For all i set $T_i = \{t \mid s \in T_i, s \to_{\mathcal{R}} t\}$
3. If $T_i = \varnothing$ for all i, then return **True**
4. If there is an i and $t \in T_i$ such that r_i is a subterm of t, then return **False**
5. Go to step 2

$\mathcal{R}_{bool}$:

$$\begin{aligned} \mathrm{and}(\mathrm{true}, \mathrm{true}) &\to \mathrm{true} \\ \mathrm{and}(x, \mathrm{false}) &\to \mathrm{false} \\ \mathrm{and}(\mathrm{false}, x) &\to \mathrm{and}(\mathrm{true}, \mathrm{not}(\mathrm{true})) \\ \mathrm{not}(\mathrm{false}) &\to \mathrm{true} \\ \mathrm{not}(\mathrm{true}) &\to \mathrm{and}(\mathrm{false}, \mathrm{false}) \end{aligned}$$

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t

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t f

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\end{aligned}$$

t f a(t, n(t))

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$$\begin{array}{ccc} \mathsf{t} & \mathsf{f} & \mathsf{a}(\mathsf{t},\mathsf{n}(\mathsf{t})) \\ & & \downarrow \\ & & \mathsf{a}(\mathsf{t},\mathsf{a}(\mathsf{f},\mathsf{f})) \end{array}$$

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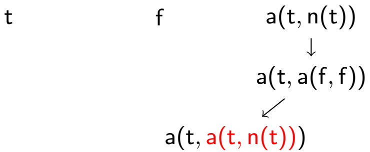

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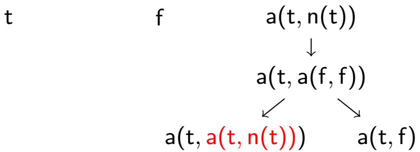

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$$\begin{tikzcd}
\mathsf{t} & \mathsf{f} & & \mathsf{a}(\mathsf{t}, \mathsf{n}(\mathsf{t})) \arrow[d] & & \mathsf{t} \\
& & & \mathsf{a}(\mathsf{t}, \mathsf{a}(\mathsf{f}, \mathsf{f})) \arrow[dl] \arrow[dr] & & \\
& & \mathsf{a}(\mathsf{t}, \textcolor{red}{\mathsf{a}(\mathsf{t}, \mathsf{n}(\mathsf{t}))}) & & \mathsf{a}(\mathsf{t}, \mathsf{f}) &
\end{tikzcd}$$

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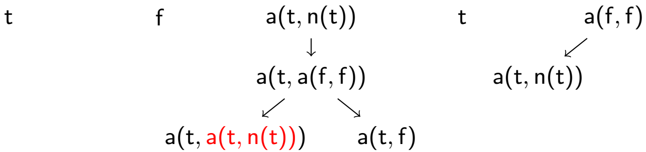

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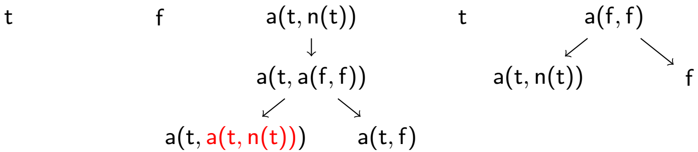

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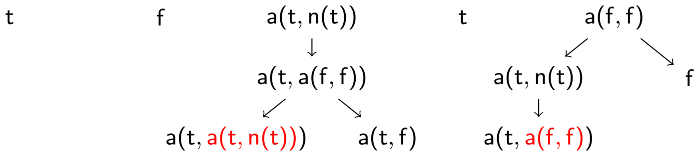

t

f

a(t, n(t)) → a(t, a(f, f)) → a(t, a(t, n(t))), a(t, f)

t

a(f, f) → a(t, n(t)), f

a(t, n(t)) → a(t, a(f, f))

Algorithm Decides Right-Ground Termination

“$\Rightarrow$:”

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Let $\mathcal{R}$ be terminating

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Let $\mathcal{R}$ be terminating
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"⇒:"

Let $\mathcal{R}$ be terminating

⇒ All rewrite sequence are finite

⇒ All search-trees T_i for all (finitely many) right-hand sides r_i are finite

⇒ Algorithm terminates after finite steps and accepts

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Let $\mathcal{R}$ be terminating
$\Rightarrow$ All rewrite sequence are finite
$\Rightarrow$ All search-trees T_i for all (finitely many) right-hand sides r_i are finite
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“$\Leftarrow$:”

Algorithm Decides Right-Ground Termination

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⇒ All rewrite sequence are finite
⇒ All search-trees T_i for all (finitely many) right-hand sides r_i are finite
⇒ Algorithm terminates after finite steps and accepts

"⇐:"

Let $\mathcal{R}$ be not terminating

Algorithm Decides Right-Ground Termination

"$\Rightarrow$:"

Let $\mathcal{R}$ be terminating
$\Rightarrow$ All rewrite sequence are finite
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"$\Leftarrow$:"

Let $\mathcal{R}$ be not terminating
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Let $\mathcal{R}$ be not terminating
$\Rightarrow_{\text{Lemma}}$ There exists a rule $\ell_i \to r_i \in \mathcal{R}$ such that $r_i \to_{\mathcal{R}}^{+} C[r_i]$
$\Rightarrow$ The search-tree T_i contains a term t such that r_i is a subterm of t after finitely many steps

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$\Rightarrow$ Algorithm terminates after finite steps and rejects

Right-Ground Almost-Sure Termination

Right-Ground PTRS

$\mathcal{R}$ is right-ground if for every rule $\ell \to \mu$ all terms $r \in \mathsf{supp}(\mu)$ are ground terms, i.e., contains no variables.

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$$\mathcal{R}_{rw}: \qquad \mathsf{g}(\mathcal{O}) \quad \to \quad \{\, {}^1\!/_2 : \mathcal{O},\ {}^1\!/_2 : \mathsf{g}(\mathsf{g}(\mathcal{O}))\,\} \qquad \text{AST}$$

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$\mathcal{R}'_{rw}: \quad g(\mathcal{O}) \to \{\, 1/3 : \mathcal{O},\ 2/3 : g(g(\mathcal{O}))\,\}$

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- How to distinguish these PTRSs?

More Restrictions!

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Tail recursive PTRS

$\mathcal{R}$ is tail recursive if for every rule $\ell \to \mu$ all rhs $r \in \mathsf{supp}(\mu)$ does not contain a defined symbol below another defined symbol.

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Tail recursive PTRS

$\mathcal{R}$ is tail recursive if for every rule $\ell \to \mu$ all rhs $r \in \mathsf{supp}(\mu)$ does not contain a defined symbol below another defined symbol.

$$\begin{array}{llcll} \mathcal{R}_{rw}: & \mathsf{g}(\mathcal{O}) & \to & \{\, {}^1\!/_2 : \mathcal{O},\ {}^1\!/_2 : \mathsf{g}(\mathsf{g}(\mathcal{O}))\,\} & \text{not TR} \\ \mathcal{R}'''_{rw}: & \mathsf{g} & \to & \{\, {}^1\!/_2 : \mathcal{O},\ {}^1\!/_2 : \mathsf{c}(\mathsf{g},\mathsf{g})\,\} & \text{TR} \end{array}$$

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$\mathcal{R}$ is non-overlapping if for all pairs of rules $\ell_1 \to \mu_1$, $\ell_2 \to \mu_2$ of $\mathcal{R}$ and all non-variable position ρ of ℓ_2 we have ℓ_1 and $\ell_2|_\rho$ not unifiable.

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$$\begin{array}{llcll}
\mathcal{R}: & \mathsf{f}(\mathsf{a}) & \to & \{\, 1 : \mathsf{f}(\mathsf{a}) \,\} & \text{not NO} \\
 & \mathsf{a} & \to & \{\, 1 : \mathsf{b} \,\} &
\end{array}$$

Decidable AST Fragment of PTRSs?

AST for right-ground, tail recursive, and non-overlapping PTRS.

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⇒: Probabilistic Petri Nets?

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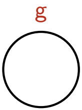

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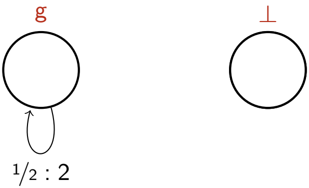

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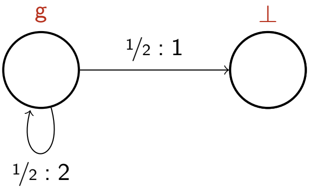

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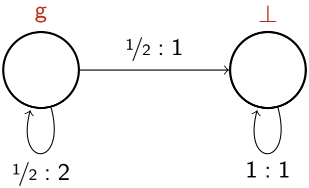

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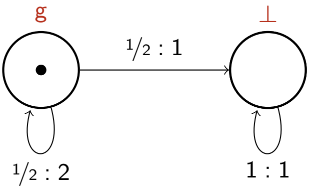

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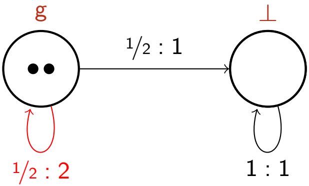

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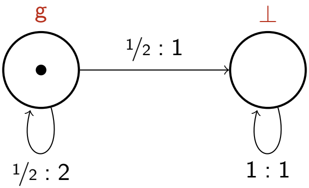

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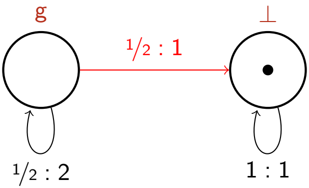

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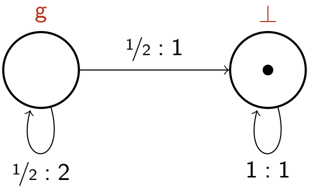

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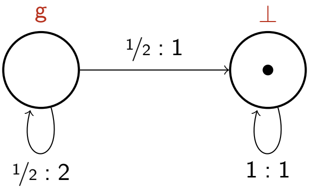

Almost-Sure Termination

For all token starts, the probability that all tokens end in ⊥ is 1.

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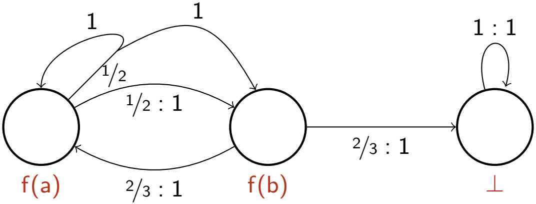
$$\mathcal{R}: \quad \begin{array}{lcl} \mathsf{f}(\mathsf{a}) & \to & \{\, {}^{1}\!/_{2} : \mathsf{c}(\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{b})),\ {}^{1}\!/_{2} : \mathsf{f}(\mathsf{b}) \,\} \\ \mathsf{f}(\mathsf{b}) & \to & \{\, {}^{1}\!/_{3} : \mathsf{a},\ {}^{2}\!/_{3} : \mathsf{f}(\mathsf{a}) \,\} \end{array}$$



Now Decidable?

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Decidable!

Open Problems:

- Can we decide AST?
    - Right-ground, tail recursive, and non-overlapping PTRS
      $\rightsquigarrow$ probabilistic petri nets?
      $\rightsquigarrow$ probabilistic grammar?
- Do we need non-overlapping?
  $\rightsquigarrow$ introduces more non-determinism
- Do we need tail-recursive?
  $\rightsquigarrow$ what do we count?
- Does it also work for PAST?

Current Research

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- DP Framework for innermost AST
    - $\mathsf{a}(\mathsf{f}, x) \to \mathsf{a}(\mathsf{t}, \mathsf{n}(\mathsf{t})) \rightsquigarrow \mathsf{A}(\mathsf{f}, x) \to \mathtt{Com}_2(\mathsf{A}(\mathsf{t}, \mathsf{n}(\mathsf{t})), \mathsf{N}(\mathsf{t}))$

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  - Non-Overlapping, Left-Linear and Non-Duplicating

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- DP Framework for (full) AST
    - Different processor definitions than for iAST

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- PTRSs with build in natural numbers
  - $\mathsf{f}(x) \to \{\frac{x}{x+1} : \mathsf{f}(x+1), \frac{1}{x+1} : \mathsf{f}(x-1)\}$

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