From Innermost To Full Almost-Sure Termination of Probabilistic Term Rewriting

Jan-Christoph Kassing, Florian Frohn, and Jürgen Giesl
RWTH Aachen

April 2024

Termination of TRSs

$\mathcal{R}_{plus}$:

$$\begin{array}{rcl} \mathsf{plus}(0, y) & \to & y \\ \mathsf{plus}(\mathsf{s}(x), y) & \to & \mathsf{s}(\mathsf{plus}(x, y)) \end{array}$$

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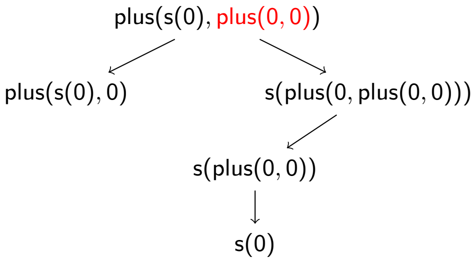

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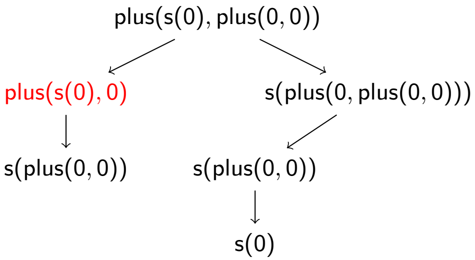

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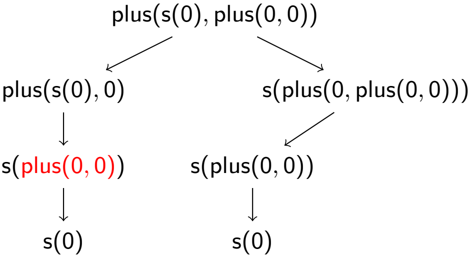

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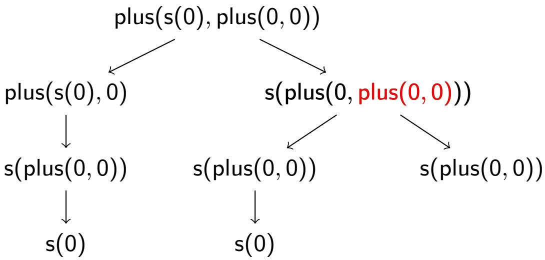

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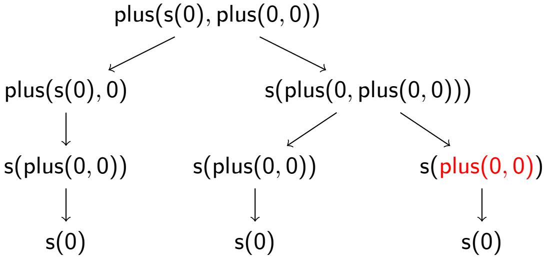

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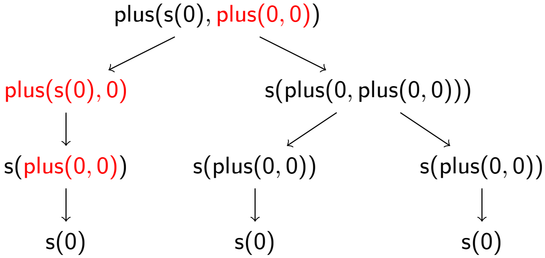

Innermost evaluation: always use an innermost reducible expression

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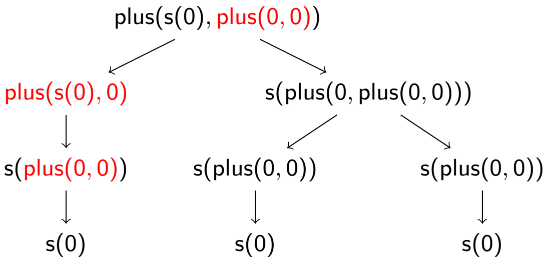

Innermost evaluation: always use an innermost reducible expression

Termination

$\mathcal{R}$ is terminating iff there is no infinite evaluation $t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \ldots$

Termination and Complexity Analysis for Programs

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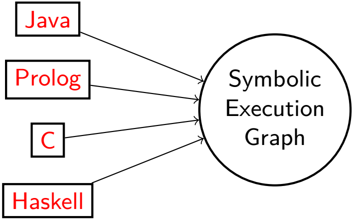
Java

Prolog

C

Haskell

Termination and Complexity Analysis for Programs



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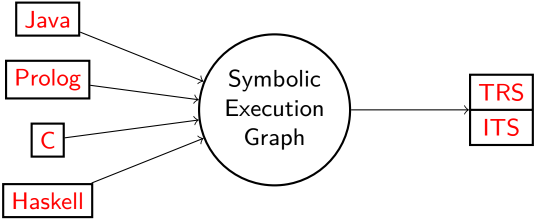

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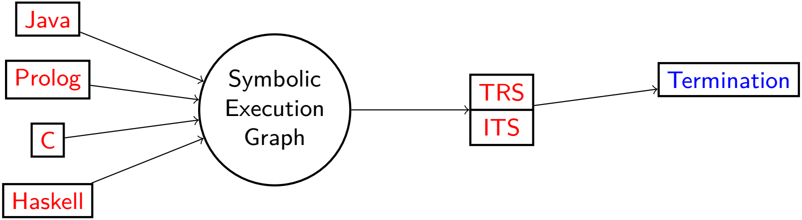

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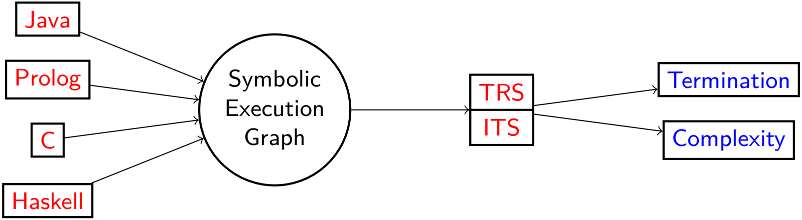

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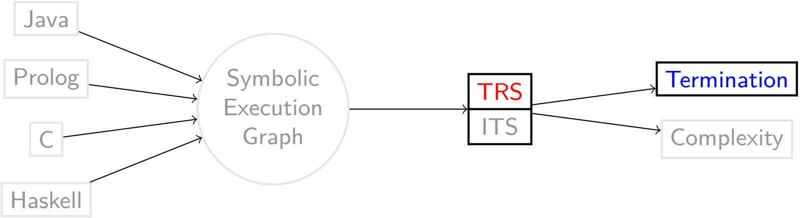

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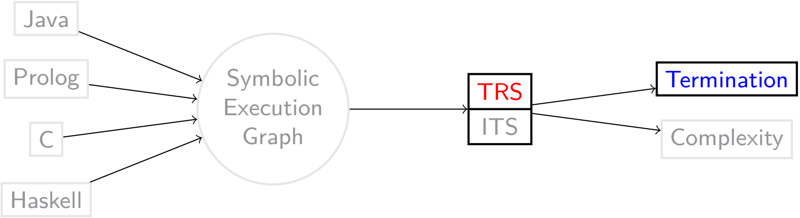

- TRS: especially good for the analysis of algorithms concerning algebraic structures and user-defined data structures
- Turing-complete programming language
  $\Rightarrow$ Termination is undecidable

Innermost Termination vs. Termination

$\mathcal{R}_2$:

$$\begin{aligned} \mathsf{f}(\mathsf{a}) &\to \mathsf{f}(\mathsf{a}) \\ \mathsf{a} &\to \mathsf{b} \end{aligned}$$

Innermost Termination vs. Termination

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Terminating? No:

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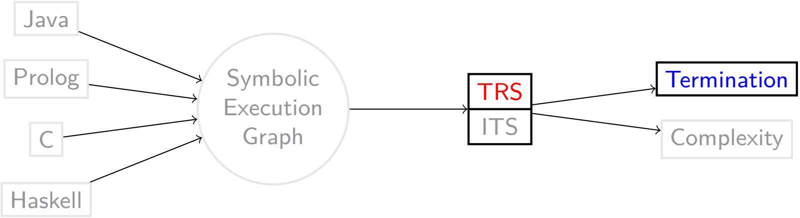
Goal: Decidable Conditions s.t. Innermost Termination $\implies$ Termination

Termination of Probabilistic TRSs



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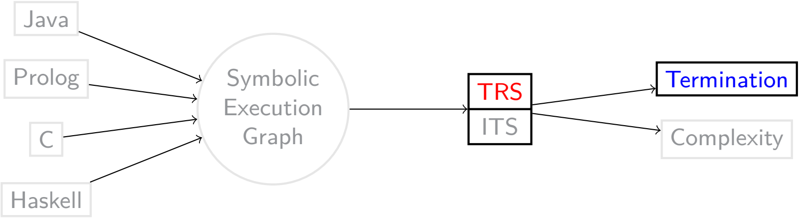

1. Relating different evaluation strategies for TRSs (non-overlapping)

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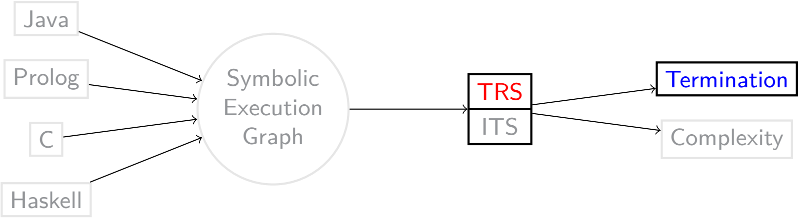

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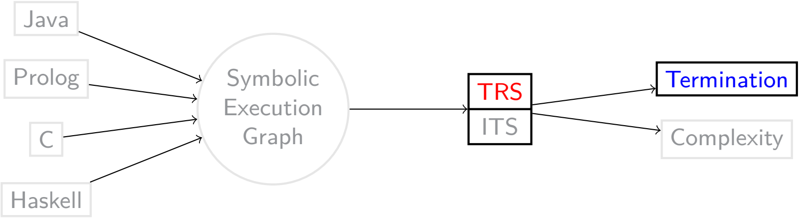

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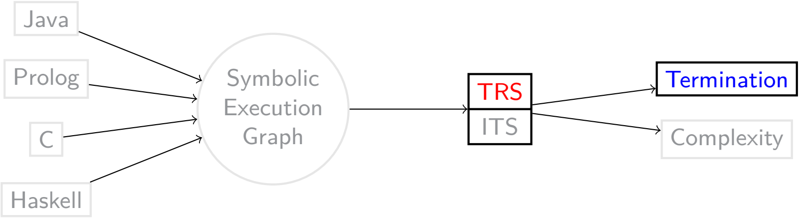

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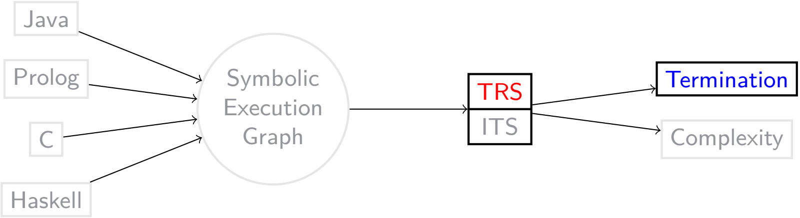

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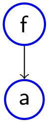

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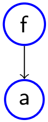

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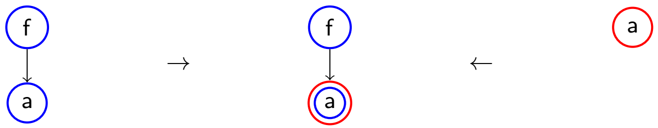

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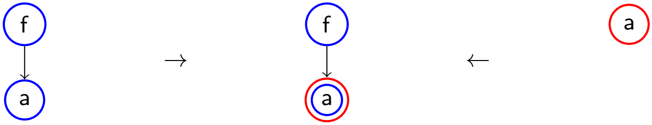

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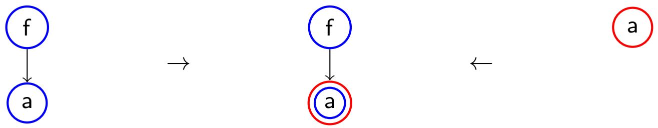

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$\to \mathcal{R}_2$ is overlapping.

Overlapping cont.

non-overlapping (Idea)

- at most one possible rewrite step at each position
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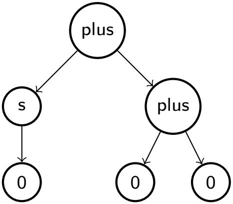

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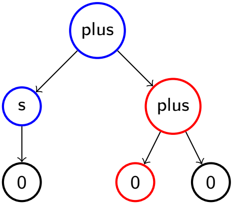

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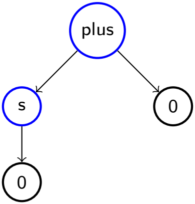

Condition for Equivalence

Theorem [Gramlich 1995]

If $\mathcal{R}$ is non-overlapping then:

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R} \text{ is innermost terminating.}$$

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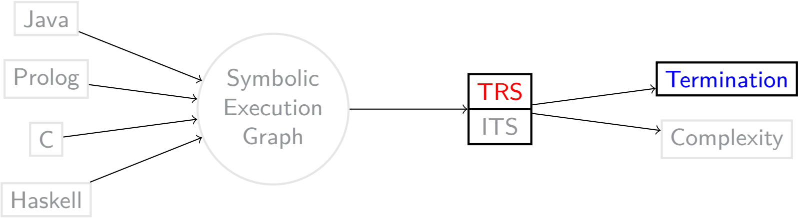

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4. Improving on left-linearity

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\mathsf{g}(0) \rightarrow \{ 1/2 : 0,\ 1/2 : \mathsf{g}(\mathsf{g}(0)) \}$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\mathsf{g}(0) \to \{\, 1/2 : 0,\ 1/2 : \mathsf{g}(\mathsf{g}(0)) \,\}$

Multi-Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\mathsf{g}(0) \quad \to \quad \{\, {}^{1}\!/_{2} : 0,\; {}^{1}\!/_{2} : \mathsf{g}(\mathsf{g}(0)) \,\}$

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$\{\, 1 : \mathsf{g}(0) \,\}$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\mathsf{g}(0) \to \{\, {}^{1}\!/_{2} : 0,\ {}^{1}\!/_{2} : \mathsf{g}(\mathsf{g}(0)) \,\}$

Multi-Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\{\, 1 : \mathsf{g}(0) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, {}^{1}\!/_{2} : 0,\ {}^{1}\!/_{2} : \mathsf{g}^2(0) \,\}$$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\mathsf{g}(0) \to \{\, 1/2 : 0,\ 1/2 : \mathsf{g}(\mathsf{g}(0)) \,\}$

Multi-Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\{\, 1 : \mathsf{g}(0) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, 1/2 : 0,\ 1/4 : \mathsf{g}(0),\ 1/4 : \mathsf{g}^3(0) \,\}$$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\quad \mathsf{g}(0) \to \{\, 1/2 : 0,\ 1/2 : \mathsf{g}(\mathsf{g}(0)) \,\}$

Multi-Distribution: $\{\, p_1 : t_1, \ldots, p_k : t_k \,\}$ with $p_1 + \ldots + p_k = 1$

$$\{\, 1 : \mathsf{g}(0) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, 1/2 : 0,\ 1/4 : \mathsf{g}(0),\ 1/4 : \mathsf{g}^3(0) \,\}$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, 1/2 : 0,\ 1/8 : 0,\ 1/8 : \mathsf{g}^2(0),$$

Termination of Probabilistic TRSs

$\mathcal{R}_{rw}$: $\mathsf{g}(0) \to \{ 1/2 : 0,\ 1/2 : \mathsf{g}(\mathsf{g}(0)) \}$

Multi-Distribution: $\{ p_1 : t_1, \ldots, p_k : t_k \}$ with $p_1 + \ldots + p_k = 1$

$$\{ 1 : \mathsf{g}(0) \}$$
$$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \}$$
$$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : 0,\ 1/4 : \mathsf{g}(0),\ 1/4 : \mathsf{g}^3(0) \}$$
$$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : 0,\ 1/8 : 0,\ 1/8 : \mathsf{g}^2(0),\ 1/8 : \mathsf{g}^2(0),\ 1/8 : \mathsf{g}^4(0) \}$$

Termination of Probabilistic TRSs

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Multi-Distribution: $\{ p_1 : t_1, \ldots, p_k : t_k \}$ with $p_1 + \ldots + p_k = 1$

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Termination for PTRSs
[Bournez & Garnier 2005, Avanzini & Dal Lago & Yamada 2019, ...]

- $\mathcal{R}$ is *terminating* iff there is no infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$

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$$\begin{aligned}
& \{\, 1 : \mathsf{g}(0) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \ & \{\, 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \ & \{\, 1/2 : 0,\ 1/4 : \mathsf{g}(0),\ 1/4 : \mathsf{g}^3(0) \,\} \\
\rightrightarrows_{\mathcal{R}_{rw}} \ & \{\, 1/2 : 0,\ 1/8 : 0,\ 1/8 : \mathsf{g}^2(0),\ 1/8 : \mathsf{g}^2(0),\ 1/8 : \mathsf{g}^4(0) \,\}
\end{aligned}$$

Termination for PTRSs
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- $\mathcal{R}$ is *almost-surely terminating (AST)* iff $\lim_{n\to\infty} |\mu_n| = 1$ for every infinite evaluation $\mu_0 \rightrightarrows_{\mathcal{R}} \mu_1 \rightrightarrows_{\mathcal{R}} \ldots$

Termination of Probabilistic TRSs

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$$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \}$$

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$$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : 0,\ 1/8 : 0,\ 1/8 : \mathsf{g}^2(0),\ 1/8 : \mathsf{g}^2(0),\ 1/8 : \mathsf{g}^4(0) \}$$

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Termination of Probabilistic TRSs

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Multi-Distribution: $\{ p_1 : t_1, \ldots, p_k : t_k \}$ with $p_1 + \ldots + p_k = 1$ $|\mu|$

$$\{ 1 : \mathsf{g}(0) \} \qquad 0$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \}$$

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Termination of Probabilistic TRSs

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$$\{ 1 : \mathsf{g}(0) \} \qquad 0$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \} \qquad 1/2$$

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	$\{ 1 : \mathsf{g}(0) \}$	0
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{ 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \}$	$1/2$
$\rightrightarrows_{\mathcal{R}_{rw}}$	$\{ 1/2 : 0,\ 1/4 : \mathsf{g}(0),\ 1/4 : \mathsf{g}^3(0) \}$	$1/2$
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Termination of Probabilistic TRSs

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$\{ 1 : \mathsf{g}(0) \}$ $\quad 0$

$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \}$ $\quad 1/2$

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Termination of Probabilistic TRSs

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$$\{ 1 : \mathsf{g}(0) \} \qquad 0$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{ 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \} \qquad 1/2$$

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Termination of Probabilistic TRSs

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$$\{\, 1 : \mathsf{g}(0) \,\} \qquad 0$$

$$\rightrightarrows_{\mathcal{R}_{rw}} \{\, 1/2 : 0,\ 1/2 : \mathsf{g}^2(0) \,\} \qquad 1/2$$

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- *positive AST (PAST)* / *strong AST (SAST)*

Innermost AST vs. AST

$\mathcal{S}_1$:

$$\mathsf{f}(\mathsf{a}) \to \{1 : \mathsf{f}(\mathsf{a})\}$$
$$\mathsf{a} \to \{1 : \mathsf{b}\}$$

AST? No:

$$\{1 : \mathsf{f}(\mathsf{a})\} \rightrightarrows_{\mathcal{S}_1} \{1 : \mathsf{f}(\mathsf{a})\} \rightrightarrows_{\mathcal{S}_1} \dots$$

Innermost AST? Yes:

$$\{1 : \mathsf{f}(\mathsf{a})\} \xrightarrow{\mathsf{i}}\!\!\!\rightrightarrows_{\mathcal{S}_2} \{1 : \mathsf{f}(\mathsf{b})\} \leftarrow \text{normal form}$$

Innermost AST vs. AST

$\mathcal{S}_1$:

$$\begin{aligned} \mathsf{f}(\mathsf{a}) &\to \{1 : \mathsf{f}(\mathsf{a})\} \\ \mathsf{a} &\to \{1 : \mathsf{b}\} \end{aligned}$$

AST? No:

$$\{1 : \mathsf{f}(\mathsf{a})\} \rightrightarrows_{\mathcal{S}_1} \{1 : \mathsf{f}(\mathsf{a})\} \rightrightarrows_{\mathcal{S}_1} \ldots$$

Innermost AST? Yes:

$$\{1 : \mathsf{f}(\mathsf{a})\} \xrightarrow{\mathsf{i}}\!\!\!\rightrightarrows_{\mathcal{S}_2} \{1 : \mathsf{f}(\mathsf{b})\} \leftarrow \text{normal form}$$

Need to restrict to non-overlapping PTRSs again

Conditions for Equivalence of AST

Does non-overlapping still suffice?

Conditions for Equivalence of AST

Does non-overlapping still suffice? $\rightarrow$ No!

Conditions for Equivalence of AST

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$\mathcal{S}_2$:

$$\begin{aligned} \mathsf{g} &\rightarrow \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\} \\ \mathsf{f}(x) &\rightarrow \{1 : \mathsf{c}(x, x)\} \end{aligned}$$

Conditions for Equivalence of AST

Does non-overlapping still suffice? $\to$ No!

$\mathcal{S}_2$:

$$\begin{array}{rcl} \mathsf{g} & \to & \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\} \\ \mathsf{f}(x) & \to & \{1 : \mathsf{c}(x, x)\} \end{array}$$

AST? No:

Conditions for Equivalence of AST

Does non-overlapping still suffice? $\to$ No!

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$$\begin{aligned} \mathsf{g} &\to \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{c}(x, x)\} \end{aligned}$$

AST? No: directly applying the f-rule to duplicate the g's:

Conditions for Equivalence of AST

Does non-overlapping still suffice? $\rightarrow$ No!

$\mathcal{S}_2$:

$$\begin{aligned} \mathsf{g} &\rightarrow \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\} \\ \mathsf{f}(x) &\rightarrow \{1 : \mathsf{c}(x, x)\} \end{aligned}$$

AST? **No**: directly applying the f-rule to duplicate the g's:

$\mathcal{S}_3$:

$$\mathsf{g} \rightarrow \{3/4 : \mathsf{c}(\mathsf{g}, \mathsf{g}), 1/4 : 0\}$$

Conditions for Equivalence of AST

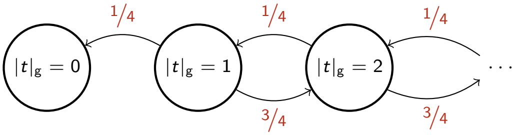
Does non-overlapping still suffice? $\to$ No!

$\mathcal{S}_2$:
$$\begin{aligned} \mathsf{g} &\to \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{c}(x, x)\} \end{aligned}$$

AST? No: directly applying the f-rule to duplicate the g's:

$\mathcal{S}_3$:
$$\mathsf{g} \to \{3/4 : \mathsf{c}(\mathsf{g}, \mathsf{g}), 1/4 : 0\}$$

$\to$ Biased random walk with $p = 3/4 > 1/2$, hence not AST.



Conditions for Equivalence of AST cont.

Innermost AST? Yes:

$$\mathsf{g} \to \{{}^{3}\!/_{4} : \mathsf{f}(\mathsf{g}), {}^{1}\!/_{4} : 0\}$$
$$\mathsf{f}(x) \to \{1 : \mathsf{c}(x, x)\}$$

Conditions for Equivalence of AST cont.

Innermost AST? Yes:

$$\mathsf{g} \to \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : \mathsf{0}\}$$
$$\mathsf{f}(x) \to \{1 : \mathsf{c}(x, x)\}$$

μ_0 :

$$\boxed{1 : \mathsf{g}}$$

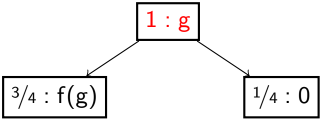
Conditions for Equivalence of AST cont.

Innermost AST? Yes:

$\mathsf{g} \to \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\}$
$\mathsf{f}(x) \to \{1 : \mathsf{c}(x, x)\}$

μ_0 :

μ_1 :



Conditions for Equivalence of AST cont.

Innermost AST? Yes:

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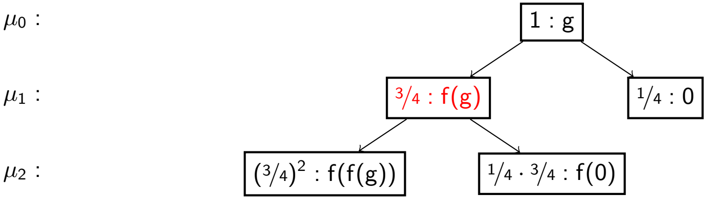
$\mathsf{f}(x) \to \{1 : \mathsf{c}(x, x)\}$

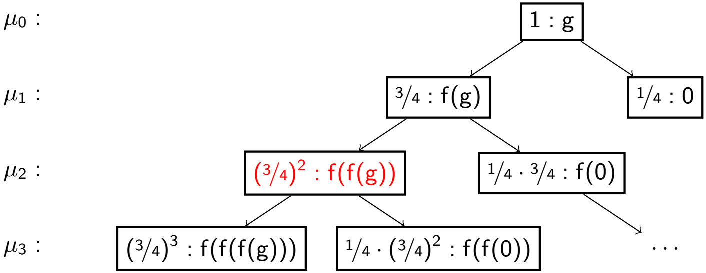


Conditions for Equivalence of AST cont.

Innermost AST? Yes:

$\mathsf{g} \to \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\}$

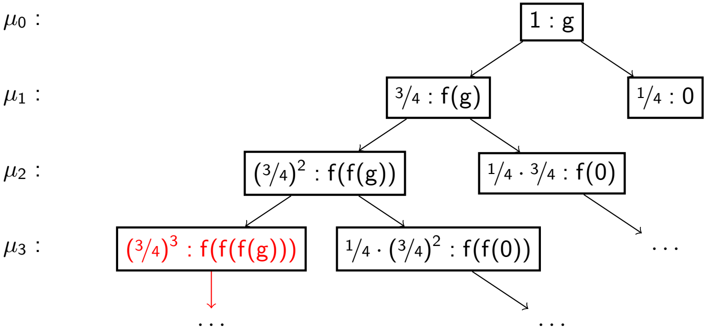
$\mathsf{f}(x) \to \{1 : \mathsf{c}(x, x)\}$



Conditions for Equivalence of AST cont.

Innermost AST? Yes:

$g \to \{3/4 : f(g), 1/4 : 0\}$
$f(x) \to \{1 : c(x, x)\}$



Conditions for Equivalence of AST cont.

Innermost AST? Yes:

$$g \to \{3/4 : f(g), 1/4 : 0\}$$
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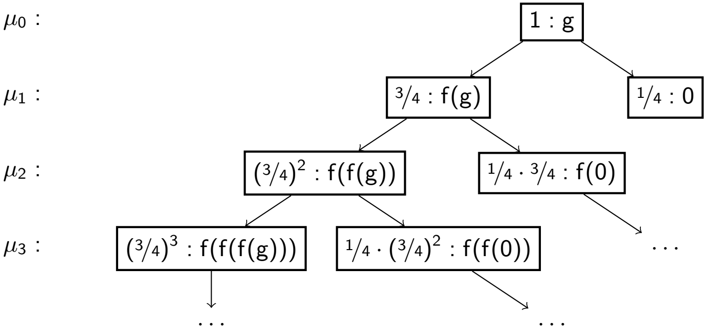

$$\lim_{k \to \infty} |\mu_k| =$$

Conditions for Equivalence of AST cont.

Innermost AST? Yes:

$$\mathsf{g} \to \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\}$$
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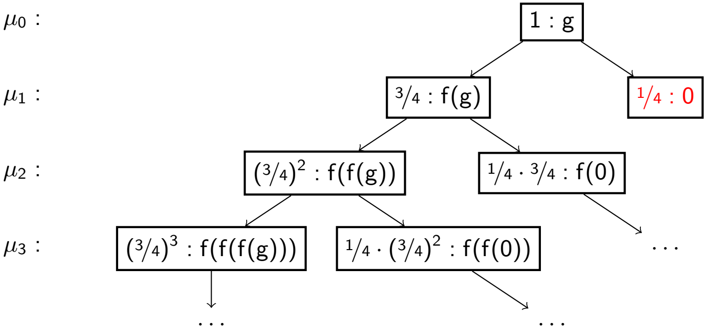
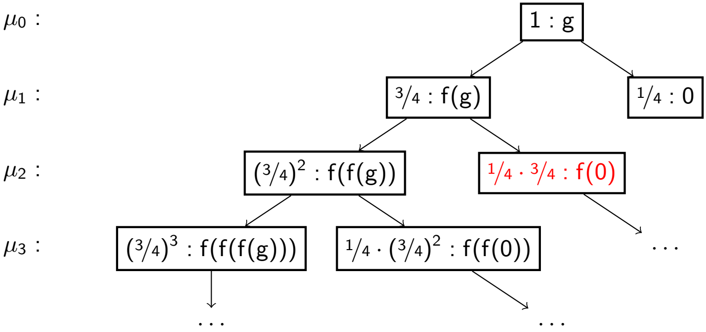


$$\lim_{k\to\infty} |\mu_k| = 1/4$$

Conditions for Equivalence of AST cont.

Innermost AST? Yes:

$\mathsf{g} \to \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\}$
$\mathsf{f}(x) \to \{1 : \mathsf{c}(x, x)\}$



$$\lim_{k \to \infty} |\mu_k| = 1/4 + 1/4 \cdot 3/4$$

Conditions for Equivalence of AST cont.

Innermost AST? Yes:

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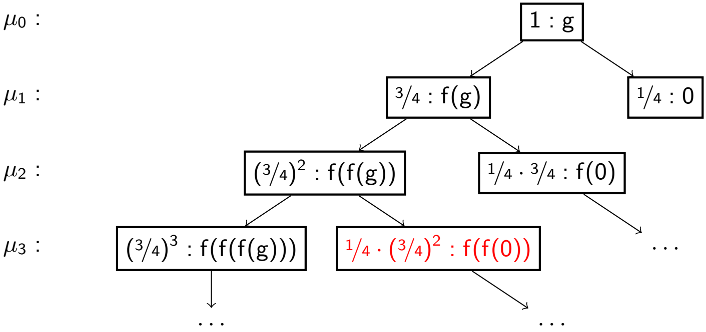
$\mathsf{f}(x) \to \{1 : \mathsf{c}(x, x)\}$



$$\lim_{k\to\infty} |\mu_k| = 1/4 + 1/4 \cdot 3/4 + 1/4 \cdot (3/4)^2$$

Conditions for Equivalence of AST cont.

Innermost AST? Yes:

$$\mathsf{g} \to \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\}$$
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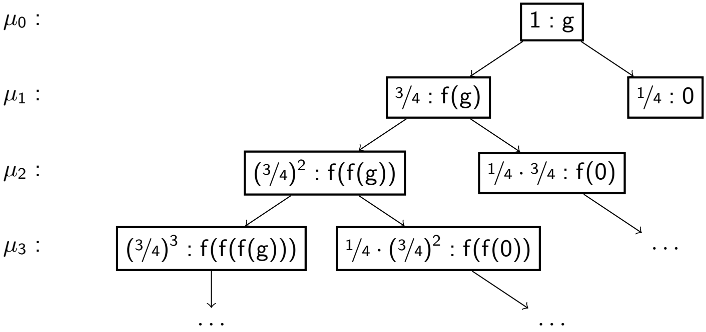

$$\lim_{k\to\infty} |\mu_k| = 1/4 + 1/4 \cdot 3/4 + 1/4 \cdot (3/4)^2 + \ldots$$

Conditions for Equivalence of AST cont.

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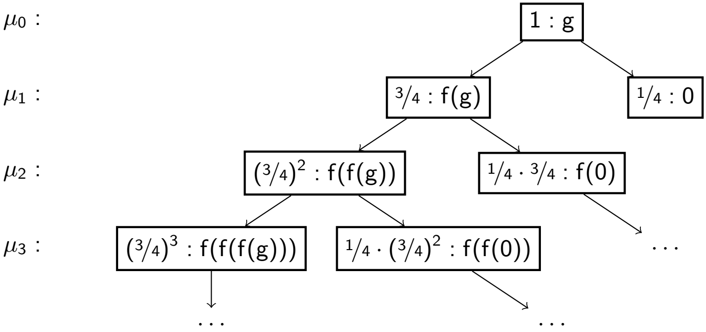
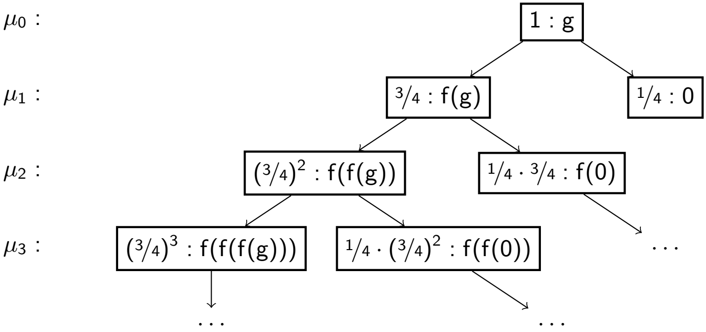


$$\lim_{k\to\infty} |\mu_k| = 1/4 + 1/4 \cdot 3/4 + 1/4 \cdot (3/4)^2 + \ldots = \sum_{i=0}^{\infty} 1/4 \cdot (3/4)^i$$

Conditions for Equivalence of AST cont.

Innermost AST? Yes:

$$g \to \{3/4 : f(g), 1/4 : 0\}$$
$$f(x) \to \{1 : c(x, x)\}$$



$$\lim_{k \to \infty} |\mu_k| = 1/4 + 1/4 \cdot 3/4 + 1/4 \cdot (3/4)^2 + \ldots = \sum_{i=0}^{\infty} 1/4 \cdot (3/4)^i = 1$$

Right-Linear

Right-Linear

Right-Linear

Right-Linear

- t is linear iff no variable occurs more than once in t
- $\{p_1 : t_1, \dots, p_k : t_k\}$ is linear iff $t_1, \dots, t_k$ are linear
- $\mathcal{S}$ is right-linear iff for all $\ell \to \mu \in \mathcal{S}$, μ is linear

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$\to$ $\mathcal{S}_2$ is not right-linear.

Conditions for Equivalence of AST cont.

Does non-overlapping and right-linear suffice?

Conditions for Equivalence of AST cont.

Does non-overlapping and right-linear suffice? $\rightarrow$ No!

Conditions for Equivalence of AST cont.

Does non-overlapping and right-linear suffice? $\rightarrow$ No!

$\mathcal{S}_4$:

$$\begin{aligned} \mathsf{a} &\rightarrow \{1/2 : \mathsf{b}, 1/2 : \mathsf{c}\} \\ \mathsf{f}(x, x) &\rightarrow \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \end{aligned}$$

Conditions for Equivalence of AST cont.

Does non-overlapping and right-linear suffice? $\to$ No!

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AST? No:

Conditions for Equivalence of AST cont.

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AST? No:

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Conditions for Equivalence of AST cont.

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AST? No:

$$\{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\}$$

Conditions for Equivalence of AST cont.

Does non-overlapping and right-linear suffice? $\rightarrow$ No!

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$$\begin{aligned} \mathsf{a} &\rightarrow \{1/2 : \mathsf{b}, 1/2 : \mathsf{c}\} \\ \mathsf{f}(x, x) &\rightarrow \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \end{aligned}$$

AST? No:

$$\{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \ldots$$

Conditions for Equivalence of AST cont.

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Innermost AST? Yes:

Conditions for Equivalence of AST cont.

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Innermost AST? Yes:

$$\{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \xrightarrow{\mathrm{i}}\!\!\!\!\rightarrow_{\mathcal{S}_4} \{1/2 : \mathsf{f}(\mathsf{b}, \mathsf{a}), 1/2 : \mathsf{f}(\mathsf{c}, \mathsf{a})\}$$

Conditions for Equivalence of AST cont.

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$$\begin{aligned} \mathsf{a} &\to \{1/2 : \mathsf{b}, 1/2 : \mathsf{c}\} \\ \mathsf{f}(x, x) &\to \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \end{aligned}$$

AST? No:

$$\{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \dots$$

Innermost AST? Yes:

$$\begin{aligned} \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} &\xrightarrow{\mathrm{i}}\!\!\!\rightrightarrows_{\mathcal{S}_4} \{1/2 : \mathsf{f}(\mathsf{b}, \mathsf{a}), 1/2 : \mathsf{f}(\mathsf{c}, \mathsf{a})\} \\ &\xrightarrow{\mathrm{i}}\!\!\!\rightrightarrows_{\mathcal{S}_4} \{1/4 : \mathsf{f}(\mathsf{b}, \mathsf{b}), 1/4 : \mathsf{f}(\mathsf{c}, \mathsf{b}), 1/4 : \mathsf{f}(\mathsf{b}, \mathsf{c}), 1/4 : \mathsf{f}(\mathsf{c}, \mathsf{c})\} \end{aligned}$$

Conditions for Equivalence of AST cont.

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AST? No:

$$\{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \ldots$$

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Conditions for Equivalence of AST

Left-Linear

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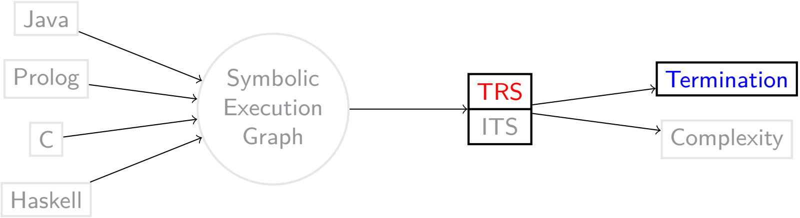
If $\mathcal{S}$ is non-overlapping, left-linear, and right-linear, then:

$$\mathcal{S} \text{ is AST} \iff \mathcal{S} \text{ is iAST}$$

Improving on Right-Linearity



1. Relating different evaluation strategies for TRSs (non-overlapping)
2. Relating different evaluation strategies for probabilistic TRSs (linear)
3. Improving on right-linearity
4. Improving on left-linearity

Spareness

$\mathcal{S}_2$:

$$\begin{aligned} \mathsf{g} &\to \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{c}(x, x)\} \end{aligned}$$

AST? No

iAST? Yes

Spareness

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AST? No
iAST? Yes

$\mathcal{S}_2'$:

$$\begin{aligned} \mathsf{g} &\to \{3/4 : \mathsf{f}(0), 1/4 : \mathsf{g}\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{c}(x, x)\} \end{aligned}$$

AST? Yes
iAST? Yes

Spareness

$\mathcal{S}_2$:

$$\begin{aligned} \mathsf{g} &\to \{\nicefrac{3}{4} : \mathsf{f}(\mathsf{g}), \nicefrac{1}{4} : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{c}(x, x)\} \end{aligned}$$

AST? No
iAST? Yes

$\mathcal{S}_2'$:

$$\begin{aligned} \mathsf{g} &\to \{\nicefrac{3}{4} : \mathsf{f}(0), \nicefrac{1}{4} : \mathsf{g}\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{c}(x, x)\} \end{aligned}$$

AST? Yes
iAST? Yes

AST? Yes:

Spareness

$\mathcal{S}_2$:

$$\begin{aligned} \mathsf{g} &\to \{{}^3/_4 : \mathsf{f}(\mathsf{g}), {}^1/_4 : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{c}(x, x)\} \end{aligned}$$

AST? No
iAST? Yes

$\mathcal{S}_2'$:

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AST? Yes
iAST? Yes

AST? Yes: directly applying the f-rule:

Spareness

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AST? No
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iAST? Yes

AST? Yes: directly applying the f-rule:

$\mathcal{S}_3'$:

$$\mathsf{g} \to \{3/4 : \mathsf{c}(0, 0), 1/4 : \mathsf{g}\}$$

Spareness

$\mathcal{S}_2$:

$$\begin{aligned} \mathsf{g} &\to \{3/4 : \mathsf{f}(\mathsf{g}), 1/4 : 0\} \\ \mathsf{f}(x) &\to \{1 : \mathsf{c}(x, x)\} \end{aligned}$$

AST? No
iAST? Yes

$\mathcal{S}_2'$:

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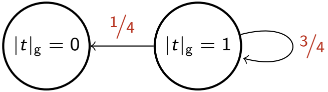
AST? Yes
iAST? Yes

AST? Yes: directly applying the f-rule:

$\mathcal{S}_3'$:

$$\mathsf{g} \to \{3/4 : \mathsf{c}(0, 0), 1/4 : \mathsf{g}\}$$

No duplication of reducible functions!



Spareness

Definition (Defined Symbols, Basic Terms)

- f is defined if there exists a rule $f(t_1, \ldots, t_n) \to \mu \in \mathcal{S}$, otherwise it is a constructor
- $f(t_1, \ldots, t_n)$ basic if f is defined and t_i only contains constructors

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- $\mathsf{plus}(0, 0)$ is basic

Spareness

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- $\mathsf{plus}(0, 0)$ is basic
- $\mathsf{plus}(\mathsf{plus}(0, 0), 0)$ is not basic

Spareness

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- $f(t_1, \ldots, t_n)$ basic if f is defined and t_i only contains constructors

Definition (Spareness)

Let $\ell \to \{p_1 : r_1, \ldots, p_k : r_k\} = \mu \in \mathcal{S}$.

- $\ell\sigma \to_{\mathcal{S}} \mu\sigma$ is spare if $\sigma(x)$ is in normal form whenever x occurs more than once in r_i
- $\mathcal{S}$ is spare if only spare rewrite steps are possible if one starts with a basic term $f(t_1, \ldots, t_n)$

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$$\{1 : \mathsf{f}(0)\} \overset{\text{(spare)}}{\rightrightarrows_{\mathcal{S}_2'}} \{1 : \mathsf{c}(0, 0)\}$$

Spareness

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$$\{1 : \mathsf{f}(0)\} \overset{\text{(spare)}}{\rightrightarrows_{\mathcal{S}_2'}} \{1 : \mathsf{c}(0, 0)\}$$

spareness + basic start term $\implies$ never duplicate defined functions.

Conditions for Equivalence of AST (3) and (4)

Thm.1

If $\mathcal{S}$ is non-overlapping, left-linear, and right-linear, then:

$$\mathcal{S} \text{ is AST} \iff \mathcal{S} \text{ is iAST}$$

Conditions for Equivalence of AST (3) and (4)

Thm.1

If $\mathcal{S}$ is non-overlapping, left-linear, and right-linear, then:

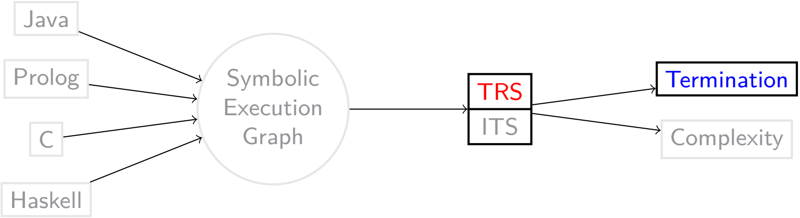
$$\mathcal{S} \text{ is AST} \iff \mathcal{S} \text{ is iAST}$$

Thm.2

If $\mathcal{S}$ is non-overlapping, left-linear, and spare, then:

$$\mathcal{S} \text{ is AST on basic terms} \iff \mathcal{S} \text{ is iAST on basic terms}$$

Improving on Left-Linearity



1. Relating different evaluation strategies for TRSs (non-overlapping)
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3. Improving on right-linearity
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Simultaneous Rewriting

$\mathcal{S}_4$:
$$\begin{aligned} \mathsf{a} &\to \{1/2 : \mathsf{b}, 1/2 : \mathsf{c}\} \\ \mathsf{f}(x, x) &\to \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \end{aligned}$$

AST? No:

$$\{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \dots$$

Innermost AST? Yes:

$$\begin{aligned} \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} &\overset{\mathsf{i}}{\rightrightarrows}_{\mathcal{S}_4} \{1/2 : \mathsf{f}(\mathsf{b}, \mathsf{a}), 1/2 : \mathsf{f}(\mathsf{c}, \mathsf{a})\} \\ &\overset{\mathsf{i}}{\rightrightarrows}_{\mathcal{S}_4} \{1/4 : \mathsf{f}(\mathsf{b}, \mathsf{b}), 1/4 : \mathsf{f}(\mathsf{c}, \mathsf{b}), 1/4 : \mathsf{f}(\mathsf{b}, \mathsf{c}), 1/4 : \mathsf{f}(\mathsf{c}, \mathsf{c})\} \end{aligned}$$

Simultaneous Rewriting

$\mathcal{S}_4$:

$$\begin{aligned} \mathsf{a} &\to \{1/2 : \mathsf{b}, 1/2 : \mathsf{c}\} \\ \mathsf{f}(x,x) &\to \{1 : \mathsf{f}(\mathsf{a},\mathsf{a})\} \end{aligned}$$

AST? **No**:

$$\{1 : \mathsf{f}(\mathsf{a},\mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \{1 : \mathsf{f}(\mathsf{a},\mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \dots$$

Innermost AST? **Yes**:

$$\begin{aligned} \{1 : \mathsf{f}(\mathsf{a},\mathsf{a})\} &\overset{\mathsf{i}}{\rightrightarrows}_{\mathcal{S}_4} \{1/2 : \mathsf{f}(\mathsf{b},\mathsf{a}), 1/2 : \mathsf{f}(\mathsf{c},\mathsf{a})\} \\ &\overset{\mathsf{i}}{\rightrightarrows}_{\mathcal{S}_4} \{1/4 : \mathsf{f}(\mathsf{b},\mathsf{b}), 1/4 : \mathsf{f}(\mathsf{c},\mathsf{b}), 1/4 : \mathsf{f}(\mathsf{b},\mathsf{c}), 1/4 : \mathsf{f}(\mathsf{c},\mathsf{c})\} \end{aligned}$$

AST w.r.t. $\rightarrowtail_{\mathcal{S}}$? **No**:

Simultaneous Rewriting

$\mathcal{S}_4$:
$$\mathsf{a} \to \{1/2 : \mathsf{b}, 1/2 : \mathsf{c}\}$$
$$\mathsf{f}(x, x) \to \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\}$$

AST? No:

$$\{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \rightrightarrows_{\mathcal{S}_4} \ldots$$

Innermost AST? Yes:

$$\{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \overset{\mathsf{i}}{\rightrightarrows}_{\mathcal{S}_4} \{1/2 : \mathsf{f}(\mathsf{b}, \mathsf{a}), 1/2 : \mathsf{f}(\mathsf{c}, \mathsf{a})\}$$
$$\overset{\mathsf{i}}{\rightrightarrows}_{\mathcal{S}_4} \{1/4 : \mathsf{f}(\mathsf{b}, \mathsf{b}), 1/4 : \mathsf{f}(\mathsf{c}, \mathsf{b}), 1/4 : \mathsf{f}(\mathsf{b}, \mathsf{c}), 1/4 : \mathsf{f}(\mathsf{c}, \mathsf{c})\}$$

AST w.r.t. $\rightarrowtail_{\mathcal{S}}$? No:

$$\{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \overset{\mathsf{i}}{\rightrightarrowtail}_{\mathcal{S}_4} \{1/2 : \mathsf{f}(\mathsf{b}, \mathsf{b}), 1/2 : \mathsf{f}(\mathsf{c}, \mathsf{c})\} \overset{\mathsf{i}}{\rightrightarrowtail}_{\mathcal{S}_4} \ldots$$

Conditions for Equivalence of AST (2)

Thm.1

If $\mathcal{S}$ is non-overlapping, left-linear, and right-linear, then:

$$\mathcal{S} \text{ is AST} \iff \mathcal{S} \text{ is iAST}$$

Conditions for Equivalence of AST (2)

Thm.1

If $\mathcal{S}$ is non-overlapping, left-linear, and right-linear, then:

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Thm.3

If $\mathcal{S}$ is non-overlapping and right-linear, then:

$$\mathcal{S} \text{ is AST} \Longleftarrow \mathcal{S} \text{ is iAST w.r.t. } \rightarrowtail_{\mathcal{S}}$$

Conditions for Equivalence of AST (2)

Thm.1

If $\mathcal{S}$ is non-overlapping, left-linear, and right-linear, then:

$$\mathcal{S} \text{ is AST} \iff \mathcal{S} \text{ is iAST}$$

Thm.3

If $\mathcal{S}$ is non-overlapping and right-linear, then:

$$\mathcal{S} \text{ is AST} \Longleftarrow \mathcal{S} \text{ is iAST w.r.t. } \rightarrowtail_{\mathcal{S}}$$

Thm.4

If $\mathcal{S}$ is non-overlapping and spare, then:

$$\mathcal{S} \text{ is AST on basic terms} \Longleftarrow \mathcal{S} \text{ is iAST w.r.t. } \rightarrowtail_{\mathcal{S}} \text{ on basic terms}$$

Conditions for Equivalence of AST (2)

Thm.1

If $\mathcal{S}$ is non-overlapping, left-linear, and right-linear, then:

$$\mathcal{S} \text{ is AST} \iff \mathcal{S} \text{ is iAST}$$

Thm.3

If $\mathcal{S}$ is non-overlapping and right-linear, then:

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If $\mathcal{S}$ is non-overlapping and spare, then:

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There exists powerful tools for iAST w.r.t. $\rightarrowtail_{\mathcal{S}}$

Implementation and Experiments

- Fully implemented in AProVE
- Evaluated on 118 benchmarks with 91 successful iAST proofs

Proofs for AST:

Old AProVE	36
Thm.1 (LL + RL)	48
Thm.3 ($\rightarrowtail_{\mathcal{S}}$+RL)	44
New AProVE	49
Thm.2 (LL+spare)	58
Thm.4 ($\rightarrowtail_{\mathcal{S}}$+spare)	56
New AProVE	61

- Arbitrary start term
- Basic start term

$$\begin{array}{lcl}
\mathsf{loop}(x) & \to & {}^1\!/\!_2 : \mathsf{loop}(\mathsf{double}(x)), {}^1\!/\!_2 : \mathsf{loop2}(x) \\
\mathsf{loop2}(\mathsf{s}(x)) & \to & 1 : \mathsf{loop2}(x) \\
\mathsf{double}(0) & \to & 1 : 0 \\
\mathsf{double}(\mathsf{s}(x)) & \to & 1 : \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))
\end{array}$$

Conclusion

- Relations between different evaluation strategies.

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$$\{1 : f(t_1, \dots, t_n)\} \rightrightarrows \dots$$

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$$\{1 : \mathsf{f}(\mathsf{a}, \mathsf{a})\} \overset{\mathsf{i}}{\rightarrowtail\!\!\!\rightrightarrows}_{\mathcal{S}_2} \{1/2 : \mathsf{f}(\mathsf{b}, \mathsf{b}), 1/2 : \mathsf{f}(\mathsf{c}, \mathsf{c})\}$$

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- We can now lift innermost AST proofs to full AST proofs.

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