Analyzing Weighted Abstract Reduction Systems via Semirings

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RWTH Aachen University

01.08.2025

Semiring Semantics in Logics

$$\psi = Ra \land (Pab \lor Pbb)$$

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a		true
b		false

P		
a	a	true
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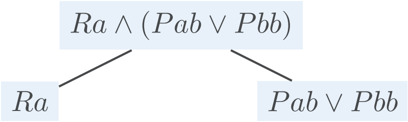

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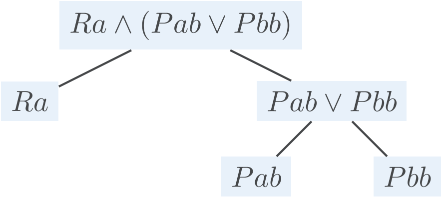

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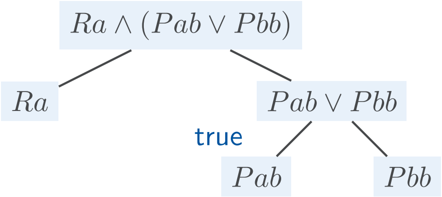

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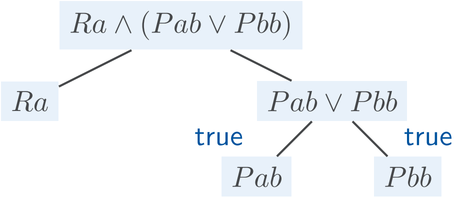

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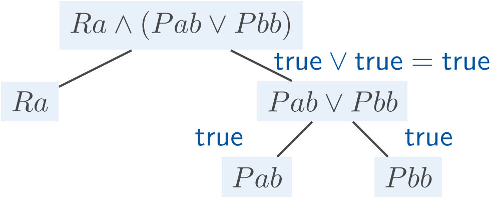

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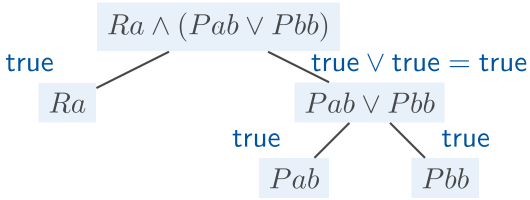

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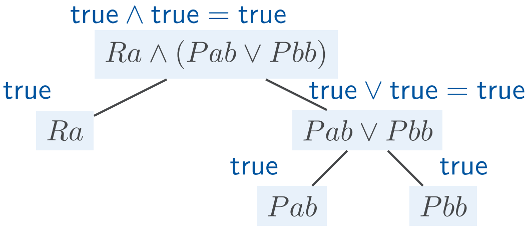

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$$\psi = Ra \wedge (Pab \vee Pbb)$$

R		cost
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b	false	∞

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a	b	true	7
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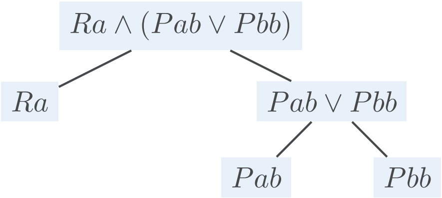

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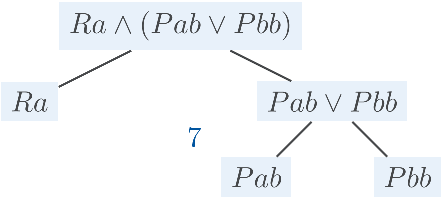

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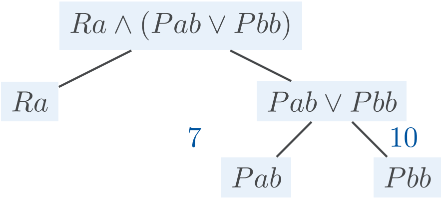

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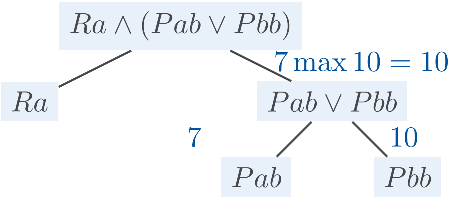

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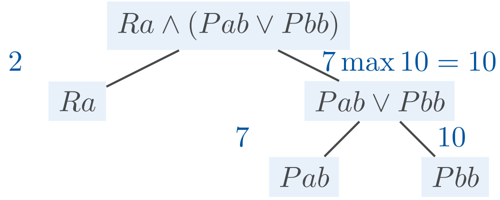

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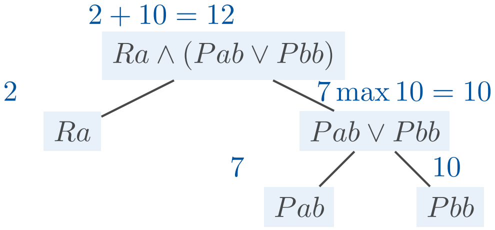

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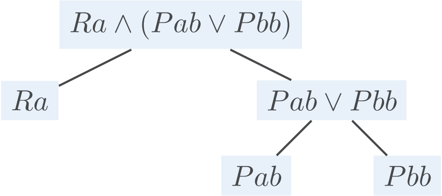

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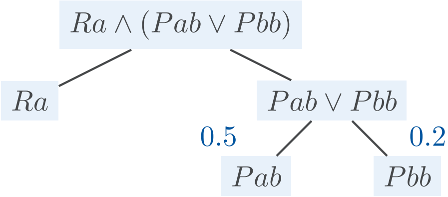

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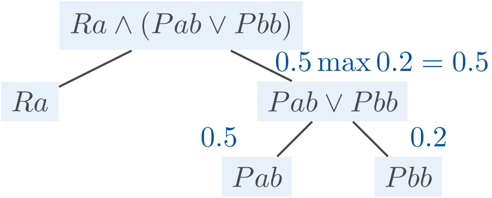

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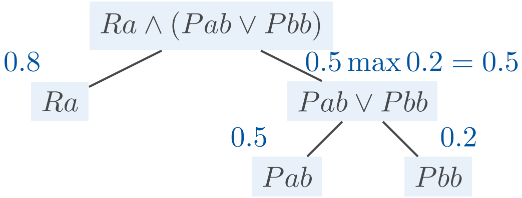

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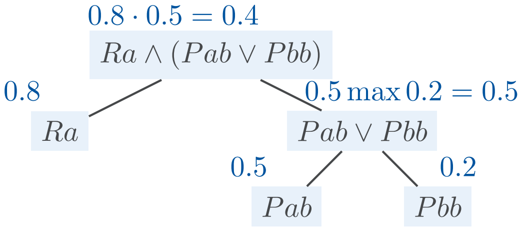

Goal

Abstract Reduction System

$$(A, \to)$$

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Weighted Rewriting

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Weighted Rewriting

Uniform Framework for Analyzing Abstract Reduction Systems

- Termination Analysis
- Safety Analysis
- Cost Analysis
- Complexity Analysis
- Arbitrary Combinations
- Many more!

Semiring

Semiring $\mathbb{S} = (S, \oplus, \odot, \mathbf{0}, \mathbf{1})$ with

- S the universe,

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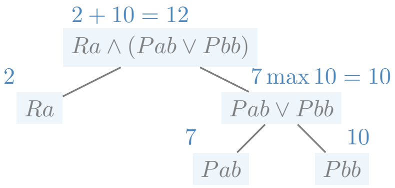
For example: $\mathbb{N} \subseteq \mathbb{N}^{\pm\infty}$ and $\sup \mathbb{N} = \top = \infty \in \mathbb{N}^{\pm\infty}$

One Framework Fits All

Weighted Abstract Reduction System $(A, \to, \mathbb{S}, \mathsf{f}_{\mathtt{NF}}, \mathsf{Aggr})$ with



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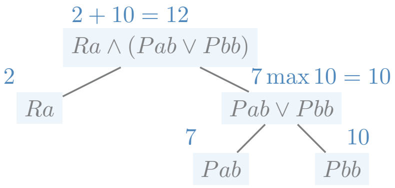
- sequence abstract reduction system $(A, \to)$,

Costs for query:

$$Ra \qquad \psi \wedge \varphi \longrightarrow [\psi, \varphi]$$

$$\psi \vee \varphi \longrightarrow [\psi, \varphi]$$



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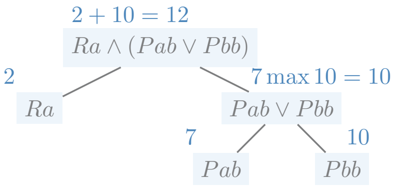

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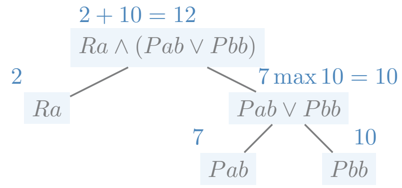

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- and aggregator functions $\mathsf{Aggr}_{a \to B}$ for each rule $a \to B$.

Costs for query: Choose $\mathbb{S}_{\mathsf{arc}} = (\mathbb{N}^{\pm\infty}, \max, +, -\infty, 0)$ and

$$\overset{2}{Ra} \qquad \overset{x+y}{\psi \wedge \varphi} \longrightarrow \overset{[x,y]}{[\psi, \varphi]}$$

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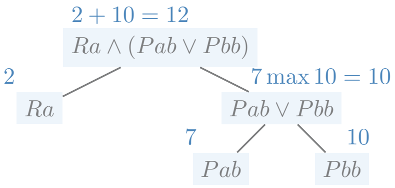

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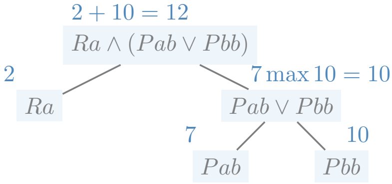

The *weight* of $a \in A$ is $[\![a]\!]$ with e.g. $[\![Ra \wedge (Pab \vee Pbb)]\!] = 12$.

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\text{false}, \text{true}\}, \vee, \wedge, \text{false}, \text{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more...

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

Sufficient Conditions for Boundedness

e.g. $\mathsf{f}_{\mathrm{NF}} \leq C$ and all $\mathrm{Aggr}_{a \to B}$ selective

Proving via Interpretations

embedding $\mathfrak{e} : A \to \mathbb{S}$

Disproving via Increasing Loops

$a \xrightarrow{+1} b \xrightarrow{+1} a$

Techniques

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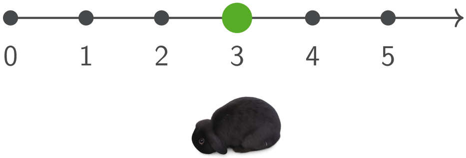

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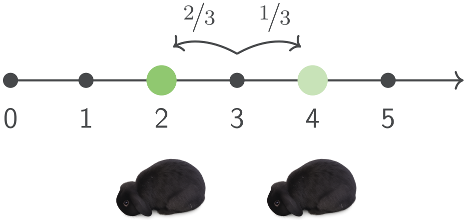

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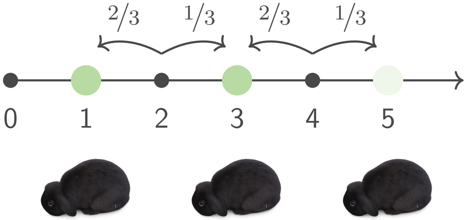

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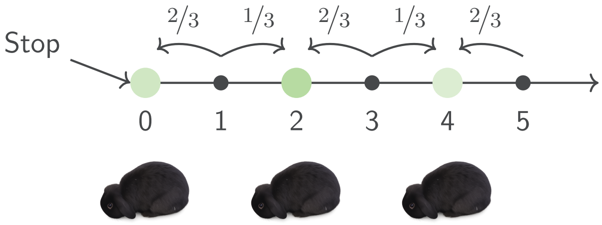

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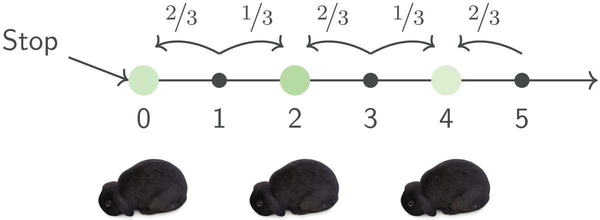

Sequence abstract reduction system $(\mathbb{N}, \to)$ with grammar

$$n + 1 \to [n, n + 2]$$

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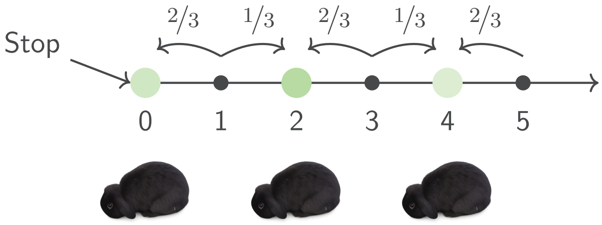

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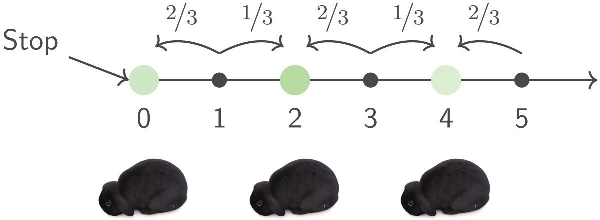

Sequence abstract reduction system $(\mathbb{N}, \to)$ with grammar

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and normal form $\mathtt{NF}_{\to} = \{0\}$.
What is the probability to reach 0 in 3 steps? What about finitely many steps?
What is the expected runtime?

Complexity of the Random Walk

Probability to reach 0 in 3 steps?

Complexity of the Random Walk

Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^\infty_{\geq 0}, +, \cdot, 0, 1)$ and

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$$0$$

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Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

$$\begin{array}{c} 1 \\ \boxed{0} \end{array}$$

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Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^\infty_{\geq 0}, +, \cdot, 0, 1)$ and

1

0 $\quad\quad$ $n+1$

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Probability to reach 0 in 3 steps? Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^\infty_{\geq 0}, +, \cdot, 0, 1)$ and

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$$\boxed{3}$$

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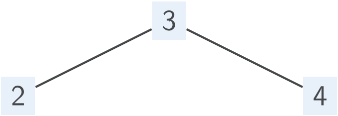

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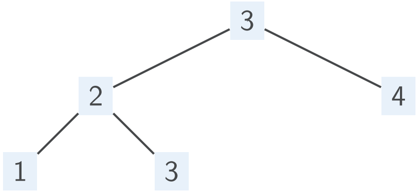

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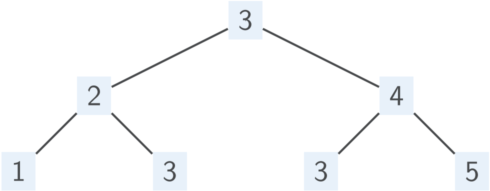

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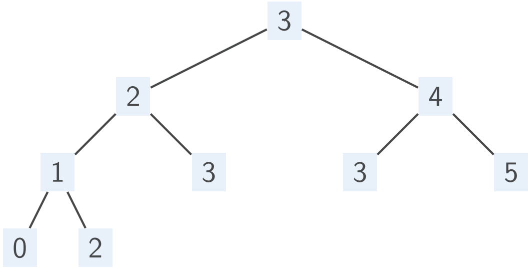

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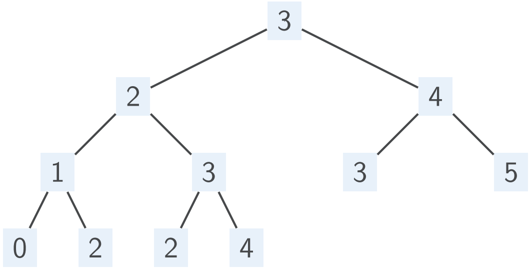

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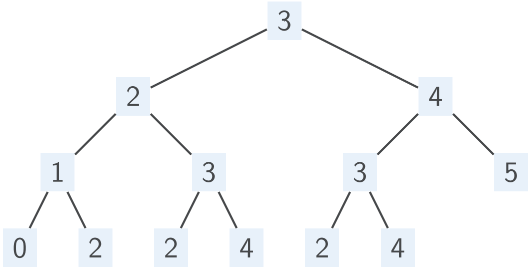

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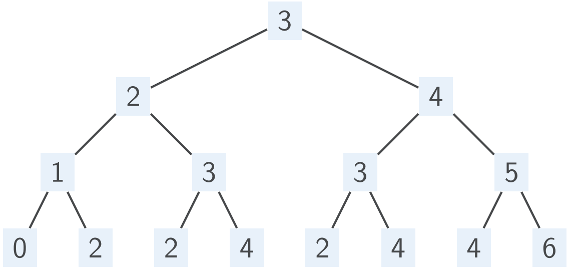

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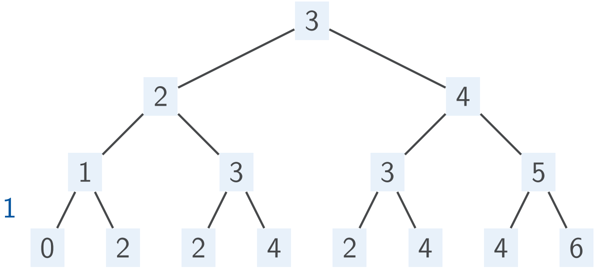

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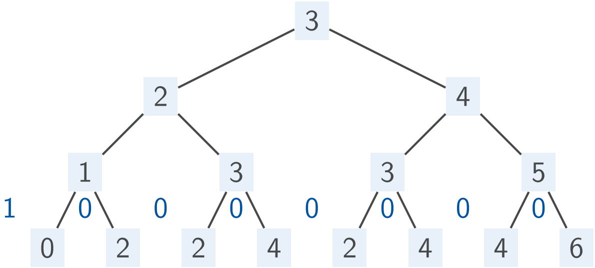

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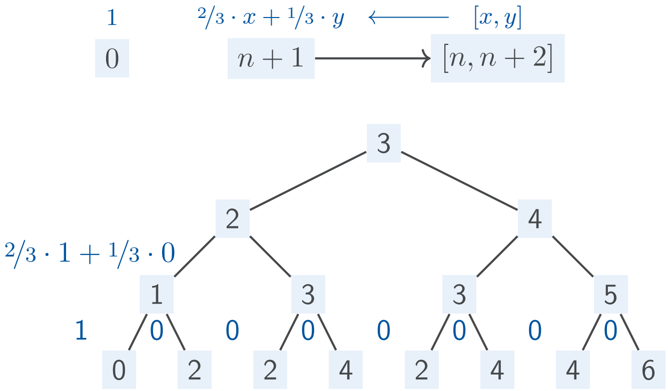

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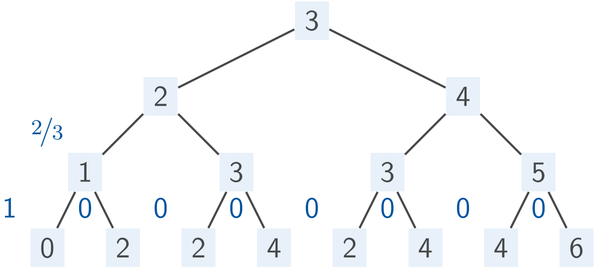

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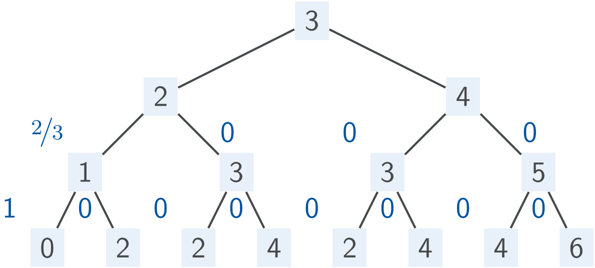

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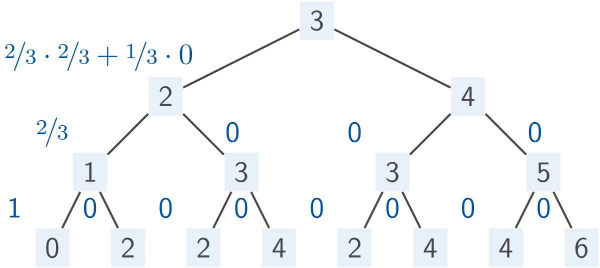

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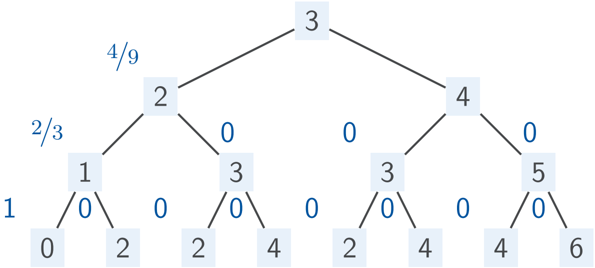

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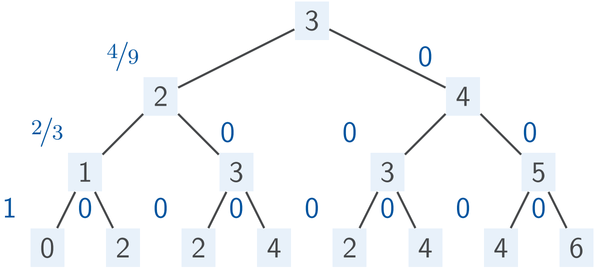

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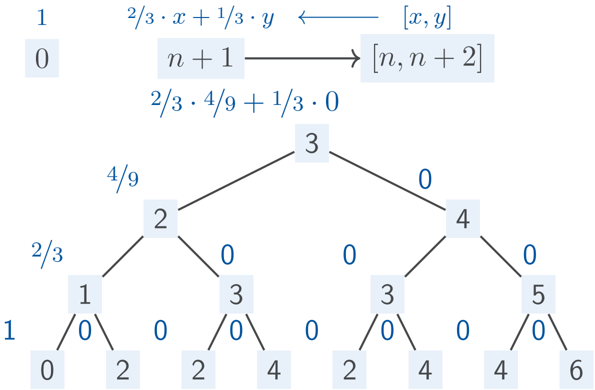

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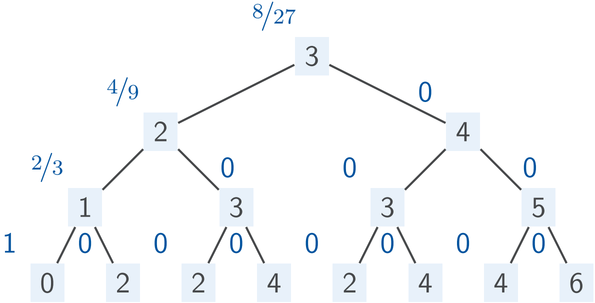

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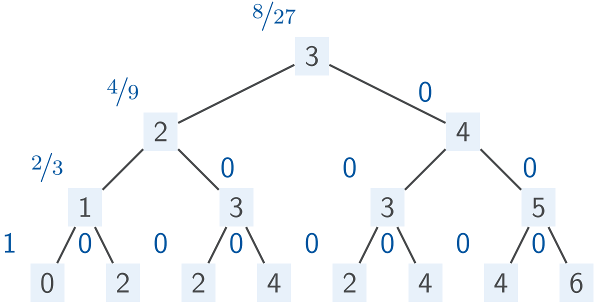

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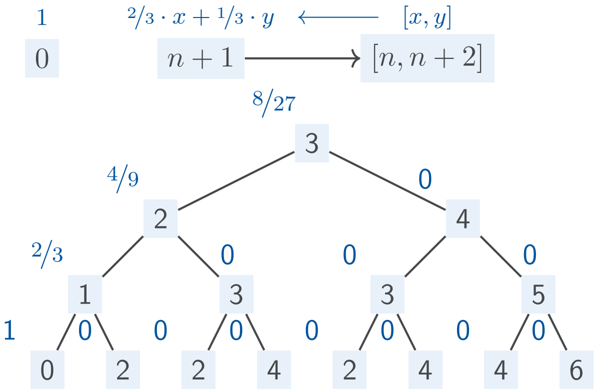
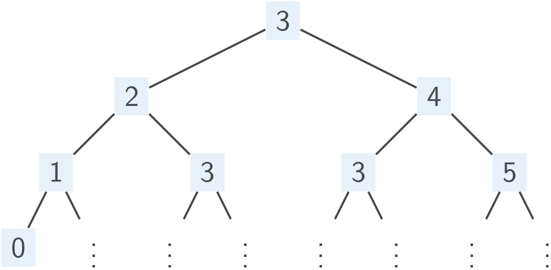


Probability to reach 0 $= \sup_{n \in \mathbb{N}} \{$ Probability to reach 0 in (at most) n steps $\}$ (Forward Unrolling)

Complexity of the Random Walk

What is the expected runtime?



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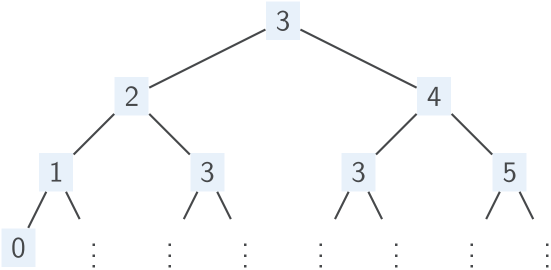

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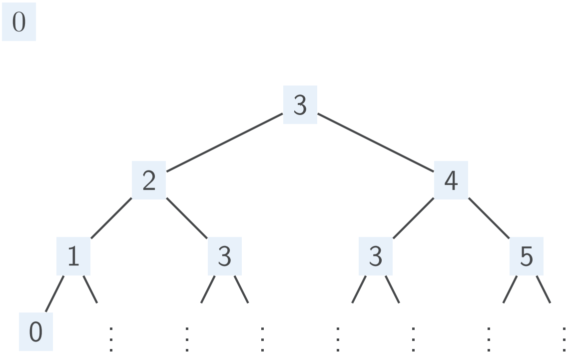

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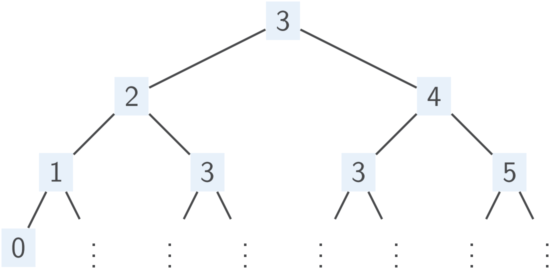

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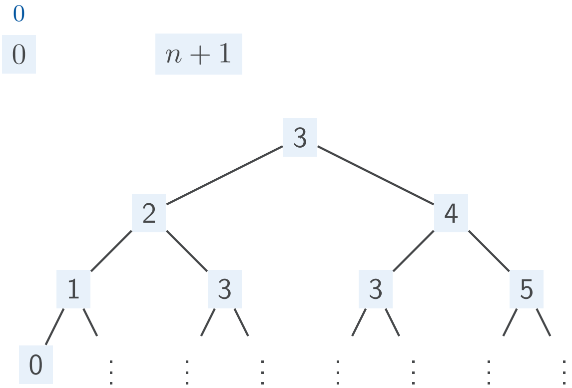

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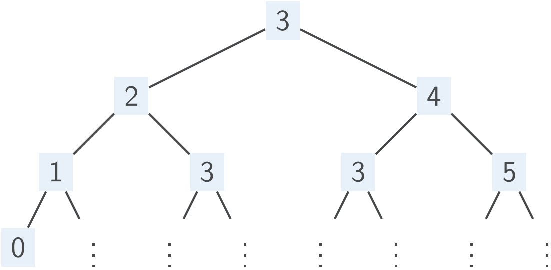

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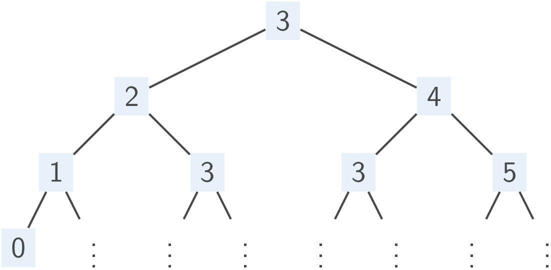

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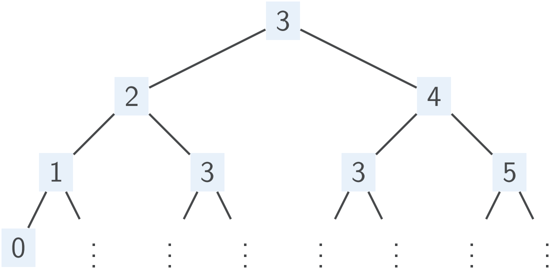

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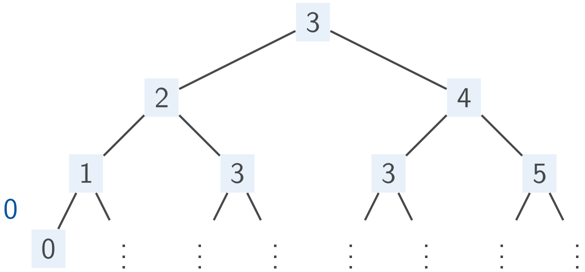

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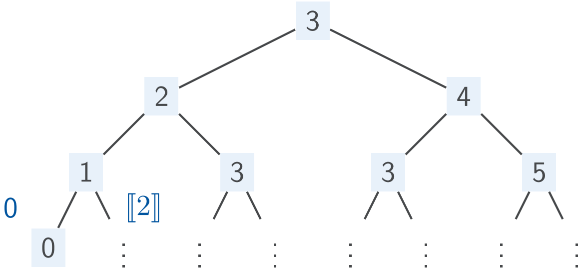

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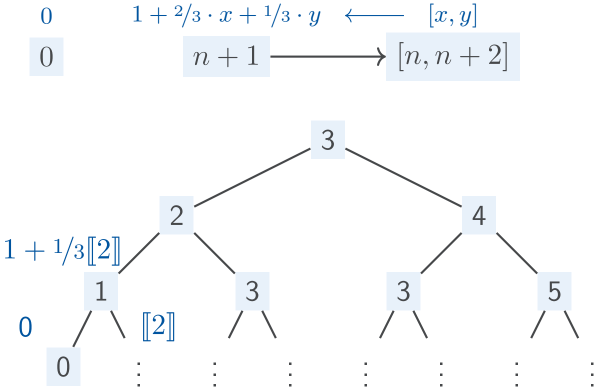

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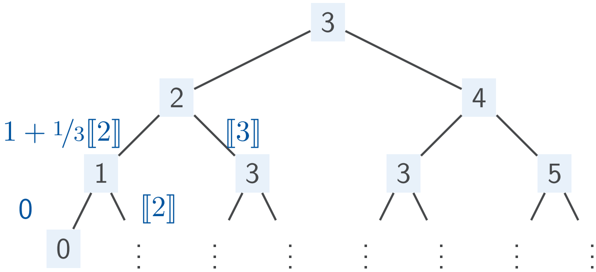

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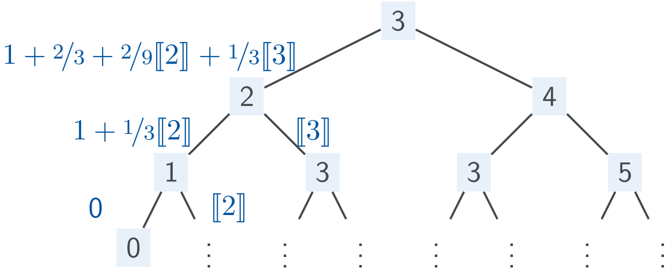

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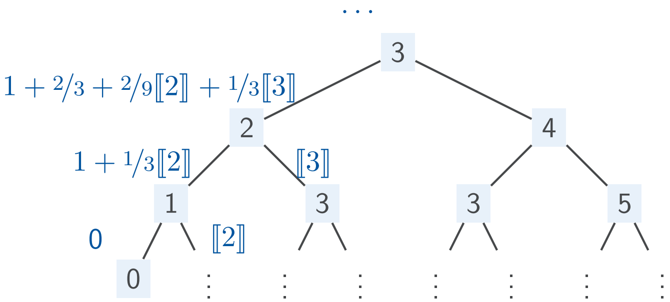

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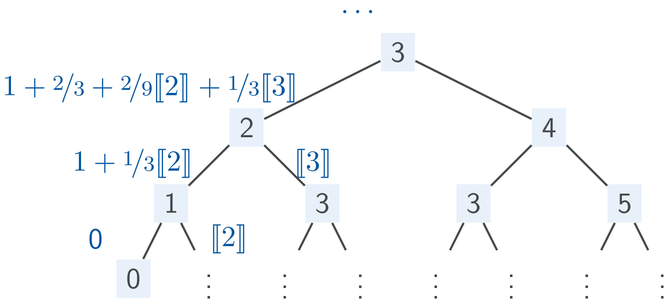
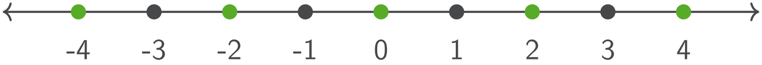


Is $[\![n]\!] \prec \infty$ for all $n \in \mathbb{N}$?

Termination and Runtime Complexity

Abstract reduction system $(\mathbb{Z}, \to)$ with

$$\begin{aligned} z \text{ is even and } z \leq -2: &\quad z \to z + 2 \\ z \text{ is even and } z \geq 2: &\quad z \to z - 2 \\ z \text{ is odd}: &\quad z \to -z \end{aligned}$$



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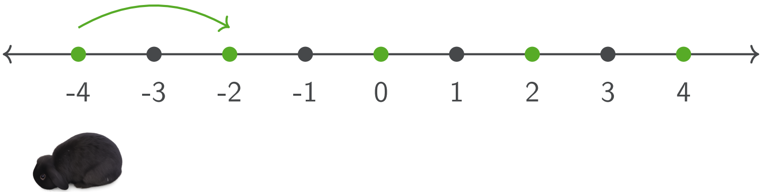

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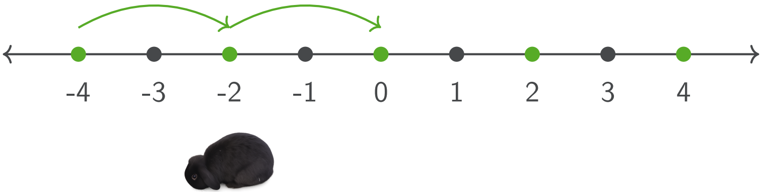

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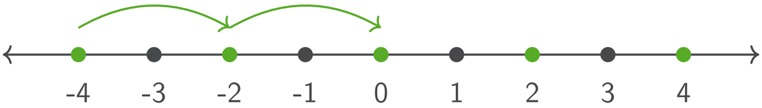

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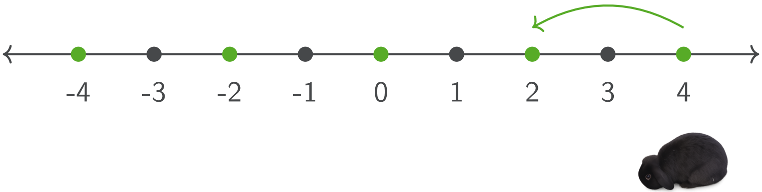

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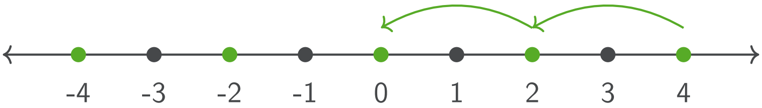

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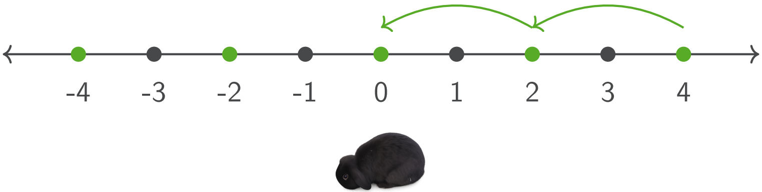

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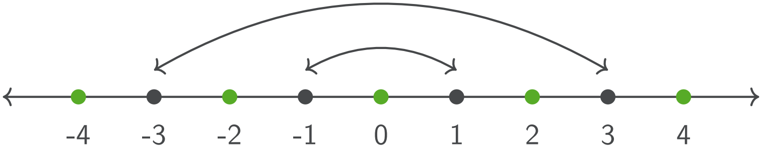

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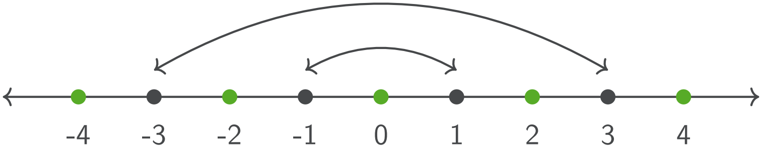

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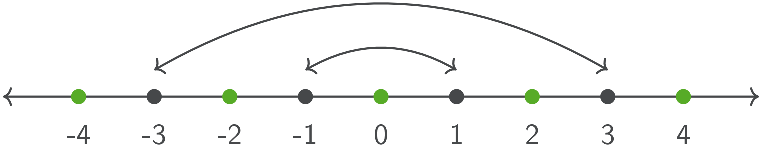

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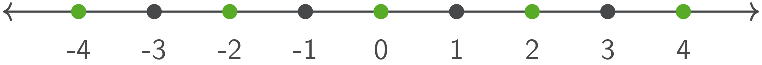

Do all reductions terminate?

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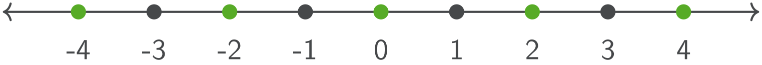

Do all reductions terminate? Choose $\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}, +, \cdot, 0, 1)$ and count steps.

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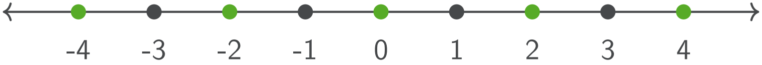

Do all reductions terminate? Choose $\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}, +, \cdot, 0, 1)$ and count steps. No!

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\text{false}, \text{true}\}, \vee, \wedge, \text{false}, \text{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more...

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

Sufficient Conditions for Boundedness

e.g. $\mathsf{f}_{\mathrm{NF}} \leq C$ and all $\mathsf{Aggr}_{a \to B}$ selective

Proving via Interpretations

embedding $\mathfrak{e} : A \to \mathbb{S}$

Disproving via Increasing Loops

$a \xrightarrow{+1} b \xrightarrow{+1} a$

Techniques

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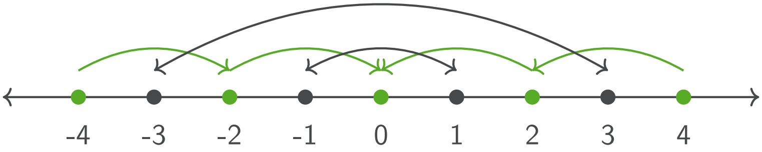

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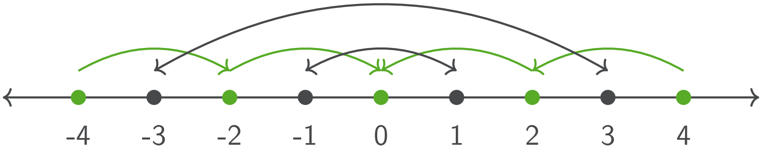

Hitting an even number is “unsafe”. Are all reductions safe?

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z is even:

z is odd:

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$\boxed{0}$ z is even: z is odd:

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$$\overset{\mathsf{true}}{\boxed{0}}$$

z is even:

z is odd:

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$$\overset{\mathsf{true}}{\boxed{0}} \qquad z \text{ is even: } \boxed{z} \qquad\qquad z \text{ is odd:}$$

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$$\overset{\mathsf{true}}{0} \qquad z \text{ is even: } \quad z \longrightarrow z \pm 2 \qquad z \text{ is odd:}$$

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$$\begin{array}{ccccc} \mathsf{true} & & \mathsf{true} \vee x & \longleftarrow & x \\ \boxed{0} & z \text{ is even:} & \boxed{z} & \longrightarrow & \boxed{z \pm 2} \quad z \text{ is odd:} \end{array}$$

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true $\quad$ $\mathsf{true} \vee x \longleftarrow x$

0 $\quad$ z is even: $\quad z \longrightarrow z \pm 2$ $\quad$ z is odd:

4

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4
|
2

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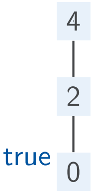

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true $\boxed{4}$

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true 2

true 0

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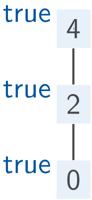

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$$\begin{array}{cc} \mathsf{true} & \boxed{4} \\ & | \\ \mathsf{true} & \boxed{2} \\ & | \\ \mathsf{true} & \boxed{0} \end{array} \qquad\qquad \boxed{5}$$

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true 4
|
true 2
|
true 0

5
|
-5

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true 4
|
true 2
|
true 0

5
|
-5
|
$[\![5]\!]$ 5
|
$\dots$

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true 4
|
true 2
|
true 0

5
|
$\mathsf{false} \vee [\![5]\!]$ -5
|
$[\![5]\!]$ 5
|
...

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$$\begin{array}{rc}
\mathsf{true} & 4 \\
 & | \\
\mathsf{true} & 2 \\
 & | \\
\mathsf{true} & 0
\end{array}
\qquad\qquad
\begin{array}{rc}
\mathsf{false} \vee [\![5]\!] & 5 \\
 & | \\
\mathsf{false} \vee [\![5]\!] & \text{-}5 \\
 & | \\
[\![5]\!] & 5 \\
 & | \\
 & \dots
\end{array}$$

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Is $[\![z]\!] < \text{true}$ for all $z \in \mathbb{Z}$?

Safety

Hitting an even number is "unsafe".

Are all reductions safe? Choose $\mathbb{S}_{\mathbb{B}} = (\{\mathsf{false}, \mathsf{true}\}, \vee, \wedge, \mathsf{false}, \mathsf{true})$ and

$$\overset{\mathsf{true}}{\boxed{0}} \qquad z \text{ is even: } \boxed{z} \xrightarrow{\mathsf{true} \vee x \;\longleftarrow\; x} \boxed{z \pm 2} \qquad z \text{ is odd: } \boxed{z} \xrightarrow{\mathsf{false} \vee x \;\longleftarrow\; x} \boxed{-z}$$

true	4	$\mathsf{false} \vee [\![5]\!]$	5
true	2	$\mathsf{false} \vee [\![5]\!]$	-5
true	0	$[\![5]\!]$	5
			…

Is $[\![z]\!] < \mathsf{true}$ for all $z \in \mathbb{Z}$? No!

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\mathsf{false}, \mathsf{true}\}, \vee, \wedge, \mathsf{false}, \mathsf{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more...

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

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e.g. $\mathsf{f}_{\mathrm{NF}} \leq C$ and all $\mathsf{Aggr}_{a \to B}$ selective

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Techniques

Combining Complexity and Safety

Are all runs terminating or safe?

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

z is even:

z is odd:

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

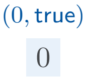
0

z is even:

z is odd:

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and



z is even:

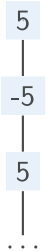
z is odd:

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$

$\boxed{0}$ z is even: $\boxed{z}$ z is odd:



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Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$

$\boxed{0}$

z is even: $\boxed{z} \longrightarrow \boxed{z \pm 2}$ z is odd:

4

2

0

5

-5

5

$\ldots$

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$\overset{(0,\mathsf{true})}{0}$ $\quad$ z is even: $\quad z \xrightarrow{\;x\;} z \pm 2$ $\quad$ z is odd:

4

2

0

5

-5

5

$\dots$

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$$\underset{0}{(0, \mathsf{true})} \qquad z \xrightarrow{(1,\mathsf{true}) \oplus x \;\longleftarrow\; x} z \pm 2$$

z is even:

$$4 \;-\; 2 \;-\; 0$$

z is odd:

$$5 \;-\; \text{-}5 \;-\; 5 \;-\; \dots$$

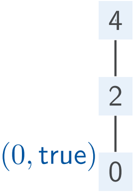
Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$

0

z is even: $\quad z \xrightarrow{(1,\,\mathsf{true}) \oplus x \;\longleftarrow\; x} z \pm 2$ $\quad z$ is odd:



Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

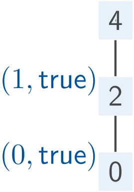
$(0, \mathsf{true})$

0

z is even: $(1, \mathsf{true}) \oplus x \longleftarrow x$

$z \longrightarrow z \pm 2$

z is odd:



Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$ $\boxed{0}$

z is even: $\boxed{z} \xrightarrow{(1, \mathsf{true}) \oplus x \;\longleftarrow\; x} \boxed{z \pm 2}$

z is odd:

$(2, \mathsf{true})$ $\boxed{4}$

$(1, \mathsf{true})$ $\boxed{2}$

$(0, \mathsf{true})$ $\boxed{0}$

$\boxed{5}$

$\boxed{-5}$

$\boxed{5}$

$\dots$

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Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

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$(2,\mathsf{true})$ 4

$(1,\mathsf{true})$ 2

$(0,\mathsf{true})$ 0

5

-5

5

. . .

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$
$\boxed{0}$

z is even: $\qquad (1, \mathsf{true}) \oplus x \longleftarrow x$

$$\boxed{z} \longrightarrow \boxed{z \pm 2}$$

z is odd:

$$\boxed{z} \longrightarrow \boxed{-z}$$

$(2, \mathsf{true})$ 4
|
$(1, \mathsf{true})$ 2
|
$(0, \mathsf{true})$ 0

5
|
-5
|
5
|
...

Combining Complexity and Safety

Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

$(0, \mathsf{true})$

0

z is even: $\qquad z \xrightarrow{(1,\mathsf{true}) \oplus x \;\longleftarrow\; x} z \pm 2$ $\qquad$ z is odd: $\qquad z \xrightarrow{x} -z$

$$\begin{array}{ll} (2,\mathsf{true}) & 4 \\ & | \\ (1,\mathsf{true}) & 2 \\ & | \\ (0,\mathsf{true}) & 0 \end{array} \qquad\qquad \begin{array}{c} 5 \\ | \\ \text{-5} \\ | \\ 5 \\ | \\ \ldots \end{array}$$

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Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

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$$\begin{array}{rc} (2,\mathsf{true}) & \boxed{4} \\ & | \\ (1,\mathsf{true}) & \boxed{2} \\ & | \\ (0,\mathsf{true}) & \boxed{0} \end{array} \qquad\qquad \begin{array}{c} \boxed{5} \\ | \\ \boxed{-5} \\ | \\ \boxed{5} \\ | \\ \ldots \end{array}$$

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$\underset{\boxed{0}}{(0, \mathsf{true})}$

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$$\begin{array}{rl} (2, \mathsf{true}) & 4 \\ & | \\ (1, \mathsf{true}) & 2 \\ & | \\ (0, \mathsf{true}) & 0 \end{array} \qquad\qquad \begin{array}{rl} & 5 \\ & | \\ (1, \mathsf{false}) \oplus [\![5]\!] & \text{-}5 \\ & | \\ [\![5]\!] & 5 \\ & | \\ & \ldots \end{array}$$

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Are all runs terminating or safe? Choose product semiring $\mathbb{S}_{\mathbb{N}^\infty} \times \mathbb{S}_{\mathbb{B}}$ and

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$$\begin{array}{rc}
(2,\mathsf{true}) & 4 \\
 & | \\
(1,\mathsf{true}) & 2 \\
 & | \\
(0,\mathsf{true}) & 0
\end{array}
\qquad\qquad
\begin{array}{rc}
(2,\mathsf{false}) \oplus [\![5]\!] & 5 \\
 & | \\
(1,\mathsf{false}) \oplus [\![5]\!] & \text{-}5 \\
 & | \\
[\![5]\!] & 5 \\
 & | \\
 & \ldots
\end{array}$$

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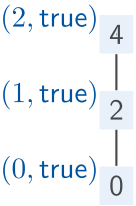

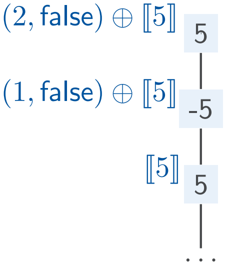

Is $[\![z]\!] < (\infty, \mathsf{true})$ for all $z \in \mathbb{Z}$?

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$(0,\mathsf{true})$ 0

$(2,\mathsf{false}) \oplus [\![5]\!]$ 5
$(1,\mathsf{false}) \oplus [\![5]\!]$ -5
$[\![5]\!]$ 5
...

Is $[\![z]\!] < (\infty,\mathsf{true})$ for all $z \in \mathbb{Z}$? Yes!

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Random Walk: Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

$$0 \qquad\qquad 1 + {}^2\!/\!_3 \cdot x + {}^1\!/\!_3 \cdot y \longleftarrow [x, y]$$

$$0 \qquad\qquad n + 1 \longrightarrow [n, n+2]$$

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Random Walk: Choose $\mathbb{S}_{\mathbb{R}^\infty} = (\mathbb{R}^{\infty}_{\geq 0}, +, \cdot, 0, 1)$ and

$$\begin{array}{ccccc} 0 & & 1 + {}^2/_3 \cdot x + {}^1/_3 \cdot y & \longleftarrow & [x, y] \\ \boxed{0} & & \boxed{n+1} & \longrightarrow & \boxed{[n, n+2]} \end{array}$$

Choose $\mathfrak{e} : n \mapsto 3 \cdot n$ such that

- $\mathfrak{e}(a) \succeq [\![a]\!] = \mathsf{f}_{\mathtt{NF}}(a)$ for all normal forms $a \in A$.
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$$\boxed{0} \qquad \boxed{n+1} \longrightarrow \boxed{[n, n+2]}$$

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$$\begin{array}{ccccc} 0 & \qquad & 1 + 2n + n + 2 & \longleftarrow & [3n, 3n+6] \\ \boxed{0} & \qquad & \boxed{n+1} & \longrightarrow & \boxed{[n, n+2]} \end{array}$$

Choose $\mathfrak{e} : n \mapsto 3 \cdot n$ such that

- $\mathfrak{e}(a) \succeq [\![a]\!] = \mathsf{f}_{\mathtt{NF}}(a)$ for all normal forms $a \in A$.
- $\mathfrak{e}(a) \succeq \mathsf{Aggr}_{a \to [b_1, \ldots, b_n]}(\mathfrak{e}(b_1), \ldots, \mathfrak{e}(b_n))$ for all $a \to [b_1, \ldots, b_n]$.

Proving via Interpretations

$(A, \to, \mathbb{S}, \mathsf{f}_{\mathtt{NF}}, \mathsf{Aggr})$ is bounded $:\Leftrightarrow$ for all $a \in A$: $[\![a]\!] \prec \top$

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Is the expected runtime finite? Yes, $[\![n]\!] \leq 3 \cdot n \prec \infty$!

Overview

Expressivity

Termination/Runtime

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Safety

$\mathbb{S}_{\mathbb{B}} = (\{\text{false}, \text{true}\}, \vee, \wedge, \text{false}, \text{true})$

Combinations

$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$

and more. . .

Prove Boundedness: $\forall a \in A : [\![a]\!] \prec \top$

Sufficient Conditions for Boundedness

e.g. $\mathsf{f}_{\mathtt{NF}} \leq C$ and all $\mathsf{Aggr}_{a \to B}$ selective

Proving via Interpretations

embedding $\mathfrak{e} : A \to \mathbb{S}$

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$a \xrightarrow{+1} b \xrightarrow{+1} a$

Techniques

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Disproving via Increasing Loops

$(A, \to, \mathbb{S}, \mathrm{f}_{\mathtt{NF}}, \mathrm{Aggr})$ is unbounded $:\Leftrightarrow$ there exists an $a \in A$: $[\![a]\!] = \top$

- Show unboundedness via induced weight polynomial $\mathcal{P}_a(X)$ of loops $a \to^+ a$

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$$\begin{array}{ccc} 0 & \qquad 1 + x \longleftarrow x & \qquad 1 + x \longleftarrow x \\ 0 & \qquad z \longrightarrow -z & \qquad z \longrightarrow z \pm 2 \end{array}$$

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$\boxed{5}$

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Are all reductions terminating? No!

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Techniques

One Framework Fits All

- Framework to unify research questions
- Lifts techniques from one area to another
- Future Work:
  - Lift more techniques to weighted rewriting
  - Confluence w.r.t. weightings
  - Automate the techniques

Abstract Reduction System

$(A, \to)$

Provenance Analysis via Semirings

$\mathbb{S}_{\mathbb{N}^\infty} = (\mathbb{N}^\infty, +, \cdot, 0, 1)$

Weighted Rewriting

Uniform Framework for Analyzing Abstract Reduction Systems

- Termination Analysis
- Safety Analysis
- Cost Analysis
- Complexity Analysis
- Arbitrary Combinations
- Many more!

Emma Ahrens, Jan-Christoph Kassing, Jürgen Giesl, Joost-Pieter Katoen:
Weighted Rewriting: Semiring Semantics for Abstract Reduction Systems.